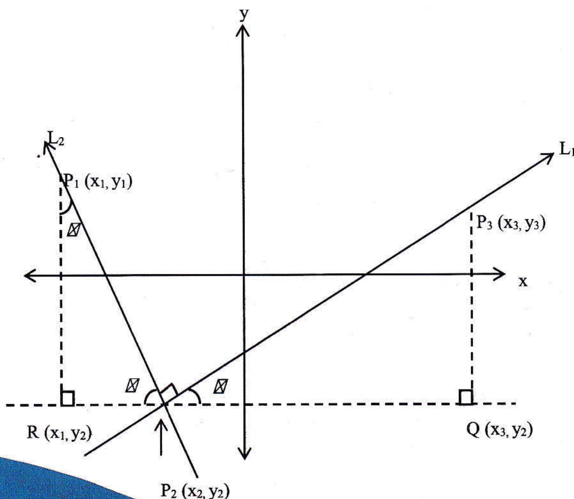
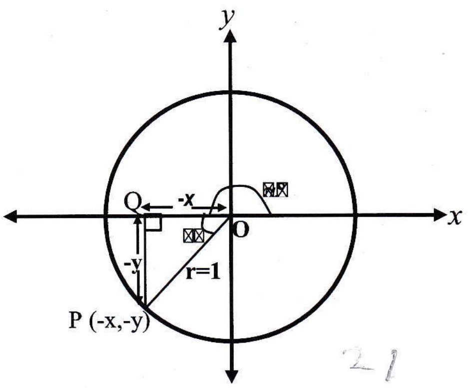
Module 5

BASIC MATHEMATICS

Stage II

Demonstrating the Use of Co-ordinate Geometry and Trigonometry





Institute of Adult Education
Alternative Secondary Education Pathway

BASIC MATHEMATICS

STAGE II

Demonstrating the Use of Co-ordinate Geometry and Trigonometry

Institute of Adult Education
Alternative Secondary Education Pathway

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Acknowledgement

This module is a product of the Institute of Adult Education (IAE) produced in 2021 for Alternative Education Pathway (AEP), and reflecting the current syllabi of 2010 secondary education in Tanzania, as recommended by Tanzania Institute of Education.

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Table of Contents

About this module

This module on Demonstrating the Use of Co-ordinate Geometry and Trigonometry has been produced by Institute of Adult Education. All modules produced by Institute of Adult Education are structured in the same way, as outlined below.

How this module is structured

The module overview

The module overview gives you a general introduction to the module. Information contained in the module overview will help you determine:

- If the module is suitable for you,
- What you already know,
- What you can expect from the module, and
- How much time you will need to invest to complete the module.

The overview also provides guidance on:

- Study skills,
- Where to get help,
- Module assignments and assessments,
- Activity icons, and
- Units.

We strongly recommend that you read the overview carefully before starting your study.

The module content

The module is broken down into units. Each unit comprises:

- An introduction to the unit content,
- Unit outcomes,
- Core content of the unit with a variety of learning activities,
- A unit reflection,
- Assignments and/or assessments, as applicable, and
- Answers to Assignment and/or assessment, as applicable.

Resources

For those interested in learning more on this subject, we provide you with a list of additional resources at the end of this module the resources are books, articles or web sites.

Your comments

After completing this Module we would appreciate it if you would take a few moments to give us your feedback on any aspect of this module. Your feedback can include comments on:

- Module content and structure,
- Module reading materials and resources,
- Module assignments,
- Module assessments, and
- Module support (assigned tutors, technical help, etc.)

Your constructive feedback will help us to improve and enhance this module.

Module overview

Welcome to this module

Dear learner, have you visited some places such as bank and bureau de change? What did you observe? Have you see some information written on the notes board? What kind of information placed on the board? Again have you ever seen things arranged in orders that follow some pattern or rules? Visit places like library, church, classroom, planned streets and other related areas and observe how things are arranged.

This module consists of four units which are Identifying Properties of Coordinates, Identifying Negative and Positive of Trigonometrically Ratios, Describing Sines and Cosines Functions and Deriving and Applying Sine and Cosine Rules and Compound Angle Formulae.

I hope you will enjoy reading this module.

General competence

Upon completion of this module you should be able to apply mathematical knowledge and skills to form lines, calculate distance between two points and solve problems on parallel and perpendicular lines in two-dimensional geometry. Furthermore, you will be able to draw graphs of sine and cosine functions and apply sine and cosine rules to solve problems that we face in the real world.

Study skills

As an adult learner' your approach to learning will be different from the approach you used during your school days. Now you will choose what you want to study, you will have professional and/or personal motivation for doing so and you will most likely be fitting your study activities around other professional or domestic responsibilities.

Essentially you will be taking control of your learning environment. As a consequence, you will need to consider performance issues related to time management, goal setting, stress management, etc. Perhaps you will also need to acquaint yourself in areas such as essay planning, coping with exams and using the web as a learning resource.

Your most significant considerations will be *time* and *space* i.e. the time you dedicate to your learning and the environment in which you engage in that learning.

We recommend that you take time now—before starting your self-study—to familiarize yourself with these issues. There are a number of excellent resources on the web. A few suggested links are:

- http://www.how-to-study.com/

  The "How to study" web site is dedicated to study skills resources. You will find links to study preparation (a list of nine essentials for a good study place), taking notes, strategies for reading text books, using reference sources, test anxiety.

- http://www.ucc.vt.edu/stdysk/stdyhlp.html

  This is the web site of the Virginia Tech, Division of Student Affairs. You will find links to time scheduling (including a "where does time go?" link), a study skill checklist, basic concentration techniques, control of the study environment, note taking, how to read essays for analysis, memory skills ("remembering").

- http://www.howtostudy.org/resources.php

  Another "How to study" web site with useful links to time management, efficient reading, questioning/ listening/observing skills, getting the most out of doing ("hands-on" learning), memory building, tips for staying motivated, developing a learning plan.

The above links are our suggestions to start you on your way. At the time of writing the web links were active. If you want to look for more go to www.google.com and type "self-study basics", "self-study tips", "self-study skills" or similar.

Need help?

Dear learner, in the course of your study, you may need help in various issues such as the location and how to get support from resource centres, clarification of various issues pertaining to your study materials (modules) and so on. If this happens, you are advised to ask for the help from your centre coordinator or facilitator. You can also visit the website of the Institute of Adult Education which is www.iae.ac.tz. You can also call no +255 22 2150838 and ask for help.

Module assessment

After each unit, you will be required to attempt one unit assignment. These are not meant for submission rather for reflection on what you have learned in the whole module. You will also be given tests and assignments for submission as you will be guided by your module facilitator. You will also sit for mock examinations to accomplish your continuous assessment.

Getting around this module

Margin icons

While working through this module you will notice the frequent use of margin icons. These icons serve to "signpost" a particular piece of text, a new task or change in activity. They have been included to help you to find your way around this module.

A complete icon set is shown below. We suggest that you familiarize yourself with the icons and their meaning before starting your study.

Activity	Assessment	Assignment	Case study
Discussion	Group activity	Help	Note it!
Outcomes	Reading	Reflection	Study skills
Summary	Terminology	Time	Tip
Computer-Based Learning	Audio	Video	Feedback
Objectives	Basic Competence	Answers to Assessments	

Unit 1

Identifying Properties of Coordinates

Introduction

Dear learner, in previous modules you learnt several concepts related to coordinate geometry when you dealt with locating point on x-y plane, solving slope of straight line forming an equation and drawing graph. In this unit you will learn more about identifying properties of co-ordinates by determining the coordinates of the midpoint of the live segment, calculate distance between two points on a plane, determining gradients of two parallel lines and describing perpendicular lines. Warmly welcome!

Learning outcomes

Upon completion of this unit you will be able to:

- Determine the coordinates of the midpoint of line segment;
- Calculate distance between two points on a plane;
- Compute gradients to determine the conditions for any two line to be parallel or perpendicular; and
- Solve problems on parallel and perpendicular lines.

Determining the Coordinates of the Midpoint of the Line Segment

Suppose you are given two points say A(5,2) and B(9,6), you can locate the points on x,y plane, join them, and find the increase in x and the increase in y.

The increase in $x = 9 - 5 = 4$

The increase in $y = 6 - 2 = 4$

Divide increase in y by 2 i.e. $\frac{4}{2}=2$ and add the results to the initial value of x i.e $5+2=7$. Divide the increase in x by 2 i.e $\frac{4}{2}$

=2, and add the result to the initial value of y i.e 2+2. You have new point (7,4). Locate the point on x,y plane. The information can be represented graphically shown in Figure 1.1.

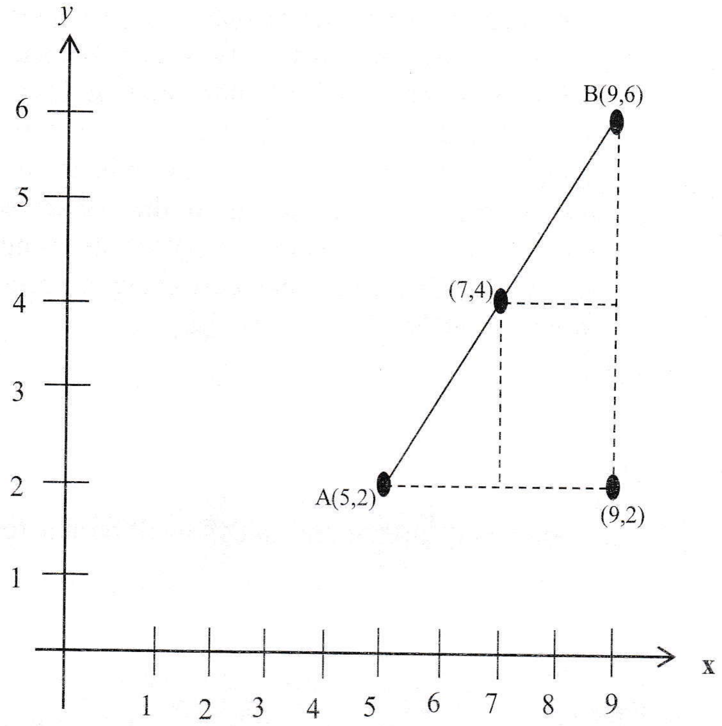


Figure1.1

Point (7,4) obtained above is the midpoint of two points A and B

Look at figure 1.2 and let P be a point with coordinates (x_1,y_1) and R with coordinate (x_2, y_2). You can find coordinates of point Q (x,y) where Q is the midpoint of PR.

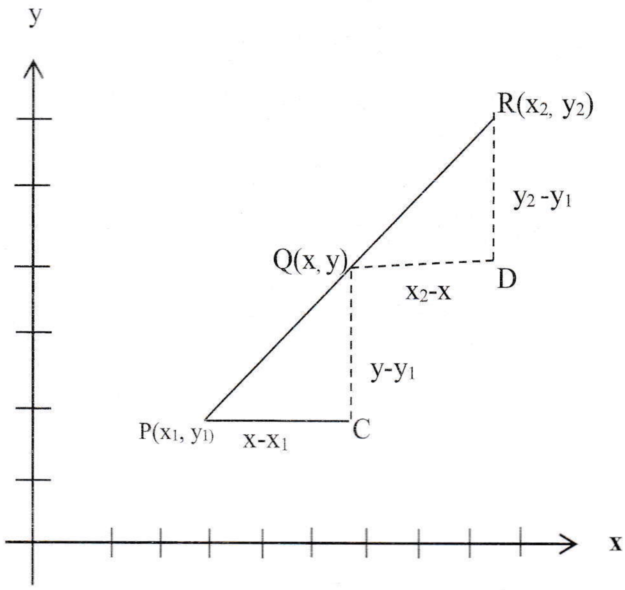


Figure 1.2

Considering angles of ΔPCQ and ΔQDR

$\Delta PCQ = \Delta QDR$ (equiangular)

Thus, $\frac{PC}{QD} = \frac{PQ}{QR} = \frac{QC}{RD}$

Using $\frac{PC}{QD} = \frac{PQ}{QR}$ gives

$$\frac{(x-x_1)}{x_2-x} = \frac{PQ}{QR}$$

But Q is the midpoint of PR

Then $\frac{PQ}{QR} = 1$

$$\therefore \frac{(x-x_1)}{x_2-x} = 1$$

Or $x - x_1 = x_2 - x$

$$2x = x_2 + x_1$$

$$X = \frac{X_2 + X_1}{2}$$

Using $\frac{PQ}{QR} = \frac{QC}{RD}$, gives

$$\frac{1}{1} = \frac{y - y_1}{y_2 - y}$$

or $\quad y_2 - y = y - y_1$

$$y_2 + y_1 = 2y$$

$$y = \frac{y_2 + y_1}{2}$$

Thus the coordinates of Q are:

$$\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Example 1

Find the coordinates of the midpoint of the line joining the points

(-4,7) and (1,-18)

Solution

The midpoint is given by:

$$\frac{(X_2 + X_1}{2} \quad \frac{y_1 + y_2)}{2}$$

$$= \left(\frac{-4+1}{2}, \frac{7+-18}{2}\right)$$

$$= \left(\frac{-3}{2}, \frac{-1}{2}\right)$$

Example 2

If the line from (-4, y_1) to (x_2, -3) is bisected at (1, -1). Find the values of y_1 and x_2

Solution

$$x = \frac{x_2 + x_1}{2}$$

$$1 = \frac{-4+x_2}{2}$$

$$2 = -4 + x_2$$

$$6 = x_2$$

From the formula

$$y = \left(\frac{y_1 + y_2}{2}\right)$$

$$-1 = \frac{y + -3}{2}$$

$$-2 = y_1 - 3$$

$$1 = y_1$$

$\therefore$ The values of x_2 and y_1 are 6 and 1 respectively.

Self Assessment

If you are given that, the vertices of the triangle are (-4, 2), (-10, -6) and (2, -3) what are the coordinates of the midpoint of its sides

Calculating Distance Between Two Points on a Plane

Consider two points P(7,2) and R(3,5) what is the distance between them? Which method do you think can be used to find distance?

Now consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ so that the distance PQ is to be determined in terms of x_1, y_1, x_2 and y_2.

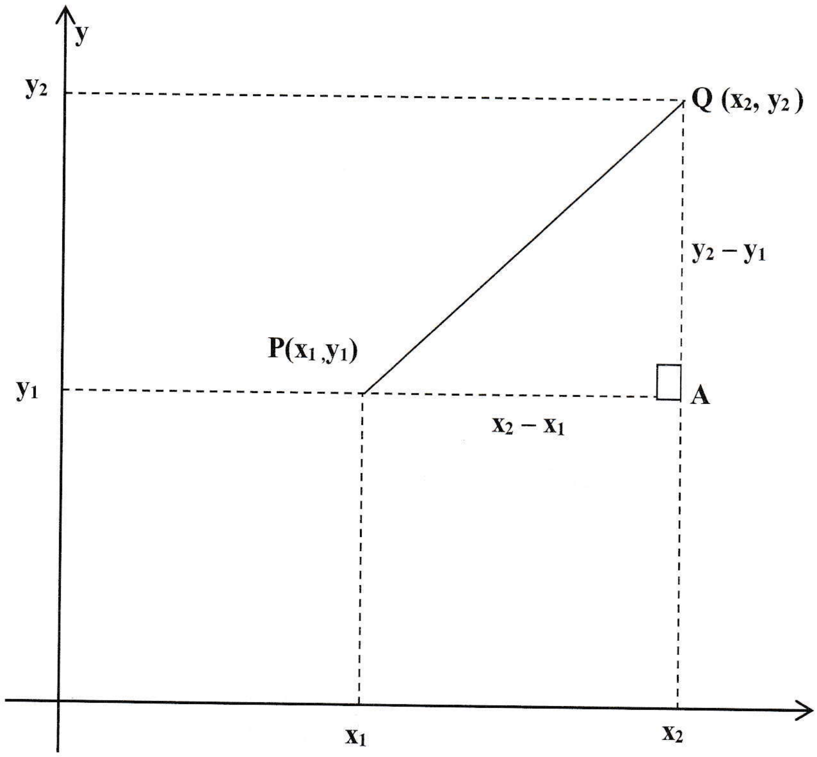


Figure 1.3

Join PQ and draw doted lines as shown in figure 1.3 Then:

$PA = x_2 - x_1$ and $AQ = y_2 - y_{1.}$

ΔPAQ is a right angled triangle.

Applying Pythagoras theorems to ΔPAQ gives

$$(PQ)^2 = (PA)^2 + (AQ)^2$$

OR $$(PQ)^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$$

$$PQ = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$

$\therefore$ Distance (d) between any two points on a plane is

obtained by the formula $d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$

Example 1.

Find the distance between (-1, 7) and (4, -5)

Solution

Taking (-1, 7) as (X_1, Y_1) and (4, -5) as (X_2, Y_2)

The required distance d is

$$d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$

$$d = \sqrt{(4 + 1)^2 + (-5 - 7)^2}$$

$$d = \sqrt{25 + 144}$$

d = 13 units

Example 2

The distance between the points (-3, 5) and $(x,2)$ is 5. Find the numerical value of x.

Solution

Taking (-3, 5) as (x,y) $(x, 2)$ as (X_2, Y_2) and distance d = 5

$$d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$

$$5 = \sqrt{(x + 3)^2 + (2 - 5)^2}$$

$$25 = (x+3)^2 + 9$$

$$16 = (x+3)^2$$

Solving the equation, the value of x is 1 or -7

Activity 1

1. Find the distance of the point (-15, 8) from the origin
2. The distance between (1, 5) and (K + 5, K + 1) is 8. Find K given that K is positive.
3. The vertices of a triangle are at (-3, -1), (2, -6) and (1, 1) show that the triangle is isosceles.

Compare your answers with the answers given at the end of this unit.

Determining Gradients of Two Parallel Lines

Dear learner, observe the house where you are living, can you identify things which are in rectangular shapes? Can you determine parallel line from those objects? In this section you are going to learn more properties of parallel lines.

Suppose L_1 and L_2 are parallel lines. Then from any point C on L_1, CED is drawn perpendicular to the x-axis, so it cuts L_2 at E and x-axis at D. as shown in figure 1.4.

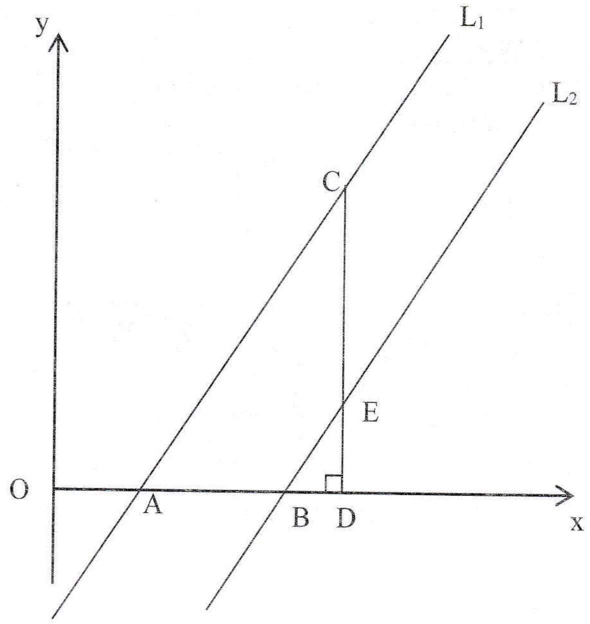


Figure 1.4

From the figure 1.4 identify equal angles. What is the relationship between $\triangle BDE$ and $\triangle ADC$?

It is not difficult for you since you learned about similar triangles in stage two.

Now it is obvious that $\triangle BDE \sim \triangle ADC$ (equiangular)

Therefore

$$\frac{DC}{AD} = \frac{DE}{BD}$$

But $\frac{DC}{AD}$ = slope of L_1 (M_1)

And $\frac{DE}{BD}$ = Slope of L_2 (M_2)

$M_1 = M_2$

If two lines are parallel, they have the same slope

Example 1

Find the equation of the line which is parallel to y=3x-7 which goes through (4,2).

Solution

Given $y = 3x - 7$ is in form of $y = mx + c$, thus the slope of this line is $M_1 = 3$

The parallel lines have the same slope.

Thus, $M_2 = M_1$

$M_2 = 3$

Required equation is $3 = \frac{y-2}{x-4}$

$3x - 12 = y - 2$

$y = 3x - 10$

Alternative solution

The line must have the same gradient as $y = 3x - 7$ which is 3.

The equation is $y = 3x + C$

Use the point (4, 2) to find c

So put $x = 4$ and $y = 2$

$2 = 3 \times 4 + C$

$-10 = C$

$\therefore$ The equation is $y = 3x - 10$

Collinear points are those point which lies on the same straight line. These are tested by choosing points on the line as common to both line segments and find their slopes. For collinear points the slopes of the segments should be equal.

Example

Show that the points A (1, 6), B (2, 4) and C (4, 0) are collinear

Collinear points have same slope, thus slope of AB=slope of BC = Slope AC

$$\frac{4-6}{2-1} = \frac{0-4}{4-2} = \frac{0-6}{4-1}$$

$$-2 = -2 = -2$$

Thus these points are collinear.

Activity 2

1. Find the equation of a straight line which passes through point A(-3, 4) and is parallel to the line with equation $3x + 4y - 15 = 0$.
2. Find the values of x so that the points A(1,3), B(-2,-3) and C(x, 7) are collinear lying on the same line.
3. A quadrilateral ABCD has vertices A(2,-3), B(2,4), C(6,6) and D(6,-6). Show that it is a trapezium with the help of graph.

Compare your answers with the answers given at the end of this unit.

Describing Perpendicular Lines

Dear learner, have you ever seen perpendicular lines in any objects around you. Hopefully you will mention many objects which are in rectangular shapes. In this section you are going to learn more properties of perpendicular lines. Let L_1 and L_2 be any two perpendicular lines with slopes M_1 and M_2 respectively as shown Figure 1.4.

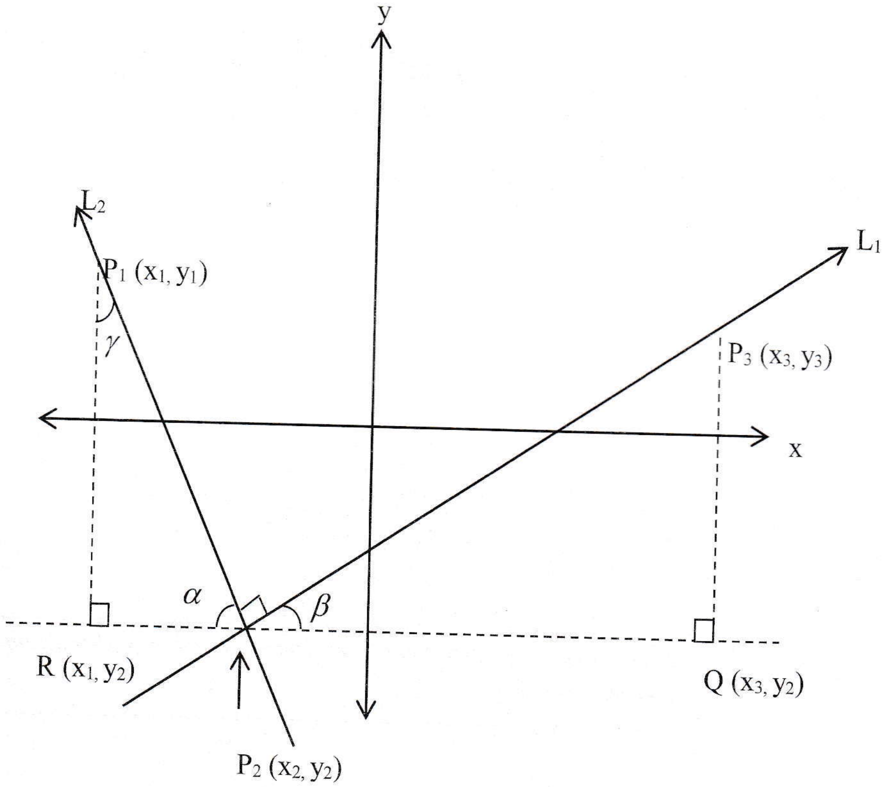


Figure 4

Choose points P_1 (x_1, y_1), $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ R and Q.

Also $\propto$, β and r are Greet letters, alpha, beta and gamma respectively, representing the degree measures of the angles as indicated.

Then

$\propto + \beta = 90°$ (complementary angles)

$\propto + r = 90°$ (complementary angles)

$\beta = r$

$\therefore \Delta P_2QP_3 \sim \Delta P_1RP_2$

$$\frac{P_2Q}{QP_3} = \frac{P_1R}{RP_2}$$

$$\frac{x_3-x_2}{y_3-y_2} = \frac{y_2-y_1}{x_2-x_1}$$

But slope of $L_1 = M_1 = \frac{y_3-y_2}{x_3-x_2}$

And slope of $L_2 = M_2 = \frac{y_2-y_1}{x_2-x_1}$

Therefore $\frac{1}{M_1} = -M_2$ or $M_1 M_2 = -1$

If two non-vertical lines are perpendicular with slopes M1 and M2, then M1 M2 = -1

Example 1

Find the equation of the line through P (-2, 5) and perpendicular to the line 6x – 7y = 4.

Solution

The equation of the line $6x - 7y = 4$

Written as $y = \frac{6}{7}x - \frac{4}{7}$

Gradient of the line is $\frac{6}{7}$

But $M_1 M_2 = -1$ then $\frac{6}{7}m_2 = -1$

$M_2 = \frac{-7}{6}$ which is the required gradient. Since the required line passes through (-2, 5) its equation is $\frac{y-5}{x+2} = \frac{-7}{6}$

$5y - 30 = -7x - 14$

$6y + 7x = 16$ or $7x + 6y - 16 = 0$

Example 2.

Show that A (-3, 2), B (5, 6) and C (7, 2) are vertices of a right angled triangle.

Solution

Slope of AB $= \frac{6-2}{5+3} = \frac{4}{8} = \frac{1}{2}$

Slope of BC $= \frac{2-6}{7-5} = \frac{-4}{2} = -2$

Since the slopes of AB and BC are negative reciprocals then triangle ABC is right angled at B.

Activity 3

1. P, Q, R and S are at (3, 1), (4, 8), (2, 10) and (m, m). If PQ and RS are perpendicular find m.

2. Show that (3, -2) lies on the line y=2x-8. Find the equation of the line through (3,-2) which is perpendicular to the line y=2x-8.

3. Find the coordinates of the foot of the perpendicular from the point (4, 3) to the line $8x+6y=25$.

4. The middle points of the adjacent sides of the quadrilateral with vertices A(5, -2), B (3, 4), C (-2, 3) and D (0,-3) are joined. By comparing gradients of the resulting figure give the geometrical identification of the figure so formed.

Compare your answers with the answers given at the end of this unit.

Unit Reflection

Now you have completed this unit. Do you think it is important to learn midpoint, distance, parallel and perpendicular lines in real life situation? What are the challenges did you face?

Suggest ways to overcome them.

Unit Assignment

1. The points A, B, C have coordinates (7, 0), (3, -3) and (-3, 3) respectively. Find the coordinates of the points Q, R, and S of the midpoint of BC, CA and AB respectively.

2. Find the coordinates of the point on the y-axis which is equidistant from the points (8, 4) and (6, 6).

3. (a) The lines $2x + 3y = 17$ and $3x - 4y=0$ meet at 1. Find the coordinates of A.

   (b) Find the equation of the line through A which is perpendicular to the line $2x+3y=17$.

   (c) Let B be at (-3, 5). Find the midpoint of $\overline{AB}$ if the coordinates of A are those in part (a).

4. Find the equations of the straight lines through the point (3,-2) which are:

   a) Parallel.

   b) Perpendicular to the line $2x + 5y = 1.7$

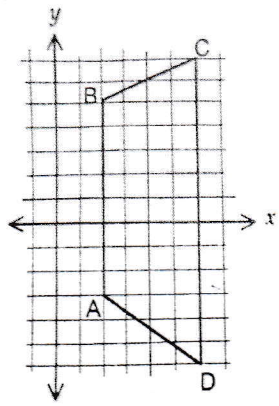
Activities Answers

Activity 1

1. 17 units
2. K= 4
3. It is Isosceles because two sides are equal with length $\sqrt{50}$ units.

Activity 2

1. 4y + 3x- 7=0
2. $x=3$
3. A trapezium is a quadrilateral having at least a pair of parallel sides as you can see that slope of AB = CD. Therefore, ABCD is a trapezium as shown in the graph.



Activity 3

1. m = 9
2. i) Lies in the line because it satisfies the equation
   ii) y=-1/2x -1
3. The coordinate of the foot is $(2, \frac{3}{2})$
4. The quadrateral formed the adjacent sides are perpendicular to each other by comparing slopes and its rectangle.

Unit 2

Identifying Negative and Positive of Trigonometric Ratios

Introduction

Dear learner, in stage I you learnt the trigonometric ratios of a unit circle on the first quadrant where, sine, cosine and tangent of angles are all positive.

In this unit you are going to study the sine, cosine and tangent of angles in all four quadrants on the unit circle and also the signs of angles on those quadrants. Therefore, at the end of this unit, you will be able to identify the quadrant in which the angles belong and you will also be able to identify signs of angles in quadrants hence you will apply the knowledge of signs of trigonometric ratios in solving various problems which are related to it.

Learning Outcomes

Upon completion of this unit you will be able to:

- Determine the sine, cosine and tangent of an angle measured in clockwise and anticlockwise directions; and
- Apply trigonometric ratios to solve problems in daily life.

Describing Signs of Trigonometric Ratio Sines, Cosines and Tangent of an Angle

Now consider an acute angle $\theta°$ $(0 \leq \theta° < 90°)$ and point P(x, y) as shown in Figure 2.1.

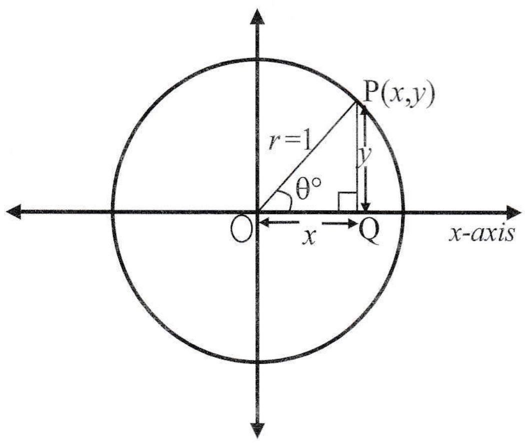


Figure 2.1

From figure 2.1, you will have the followings:

(i) $\text{Sin } \theta° = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{PQ}{OP} = \dfrac{y}{r}$

But r = 1, since the radius OP = 1

$\therefore \text{Sin } \theta° = \dfrac{y}{1} = y$

$\therefore \text{Sin } \theta° = y$ (positive).

(ii) $\cos \theta° = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{OQ}{OP} = \dfrac{x}{r}$

But $r = 1$, since the radius OP = 1

$\therefore \cos \theta° = \dfrac{x}{1} = x$

$\therefore \cos \theta° = x$ (positive).

(iii) $\tan \theta° = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{y}{x}$

$\therefore \tan \theta° = \dfrac{y}{x}$ (positive)

Therefore, you will see that all the sine, cosine and tangent of angles in the first quadrant are positive, as you have seen in the above examples on figure 2.1 as shown by point P (x, y) on the first quadrant.

Acute angle is an angle which is greater than 0° but less than 90o

$0° < \theta° < 90°$

$\theta°$ *- is an acute angle.*

2. Consider an obtuse angle (a°): i.e $90° < \alpha° < 180°$ and the point P $(-x, y)$ as shown by figure 2.2.

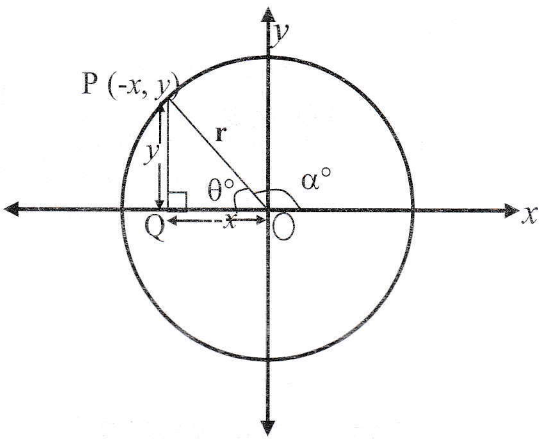


Figure 2.2

From figure 2.2, you can see that $\alpha°$ is an obtuse angle that is an angle which is greater than $90°$ but less than $180°$

$(90° < \alpha° < 180°)$, and it is in the second quadrant.

Note: $\theta° = 180° - \alpha°$

Then:

(i) $\sin\theta° = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{PQ}{OP} = \frac{x}{r}$

but $r = 1$

$\therefore \sin\theta° = \frac{y}{1} = y$

$\therefore \sin\theta° = y$ (positive)

(ii) $\cos\theta° = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{OQ}{OP} = -\frac{x}{r}$

but $r = 1$, since the radius OP = 1

$\therefore \cos\theta° = -\frac{x}{1} = -x$

$\therefore \cos\theta° = -x$ (negative)

(iii) $\tan\theta° = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{PQ}{OQ} = \frac{y}{x}$

$\therefore$ **tan** $\theta° = \frac{y}{x}$ (negative)

Therefore, in second quadrant only sine of angles is positive. The cosine and the tangent of angles in this quadrant are negative.

3. Consider the reflex angle a° in the third quadrant (180°< θ°<270°) and point P ($-x$, $-y$) as shown in the Figure 2.3 below.

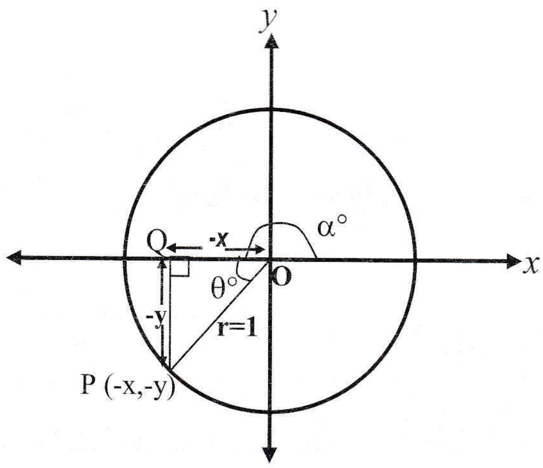


Figure 2.3

From Figure 2.3 α° is a reflex angle, that is an angle between 180° and 270° and it is in the third quadrant.

Note: $\theta° = \alpha° - 180°$

Then:

(i) $\sin\theta° = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{PQ}{OP} = -\frac{y}{r}$

but $r = 1$

$\sin\theta° = -\frac{y}{1}$,

$\therefore \sin\theta° = -y$ (negative).

(ii) $\cos\theta° = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{OQ}{OP} = -\frac{x}{r}$

but r = 1

$\cos\theta° = -\frac{x}{1} = -x$

$\therefore \cos\theta° = -x$ (negative)

(iii) $\tan\theta° = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{PQ}{OQ} = \frac{-y}{x}$

$\therefore \tan\theta° = \frac{-y}{x}$ (negative)

Therefore, in the third quadrant only tangent is positive. But sine and cosine of angles in the third quadrant are negative.

4. Let us consider the Reflex angle $\alpha°$ which is in the fourth quadrant as shown in figure 2.4.

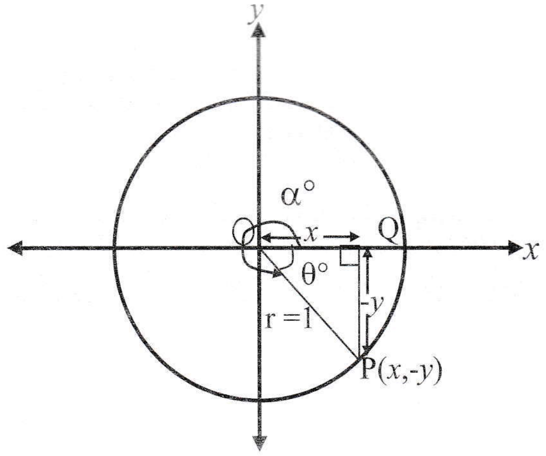


Figure 2.4

From Figure 2.4, point P(x, $-y$) is in the fourth quadrant and angle α° is a reflex angle (270 < α° < 360°).

Note: $\theta° = 360° - \alpha°$

Then;

(i) $\sin\theta° = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{PQ}{OP}$ P $= -\dfrac{y}{r}$

But $r = 1$

$\therefore \sin\theta° = -\dfrac{y}{1} = -y$

$\therefore \sin\theta° = -y$ (negative)

(ii) $\cos\theta° = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{OQ}{OP} = \dfrac{x}{r}$

But $r = 1$

$\therefore \cos\theta° = \dfrac{x}{1} = x$

$\therefore \cos\theta° = x$ (positive)

(iii) $\tan \theta° = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{PQ}{OQ} = -\dfrac{y}{x}$

$\therefore \tan \theta° = -\dfrac{y}{x}$ (negative)

Therefore, in the fourth quadrant only cosine is positive. But the sine and tangent are negative.

You have seen that the trigonometric ratios are positive or negative depending on the size of the angles and the quadrant in which the angles are found.

The results obtained can be summarized in order to help you in determining whether sine, cosine and tangent of an angle is positive or negative as. See Figure 2.5

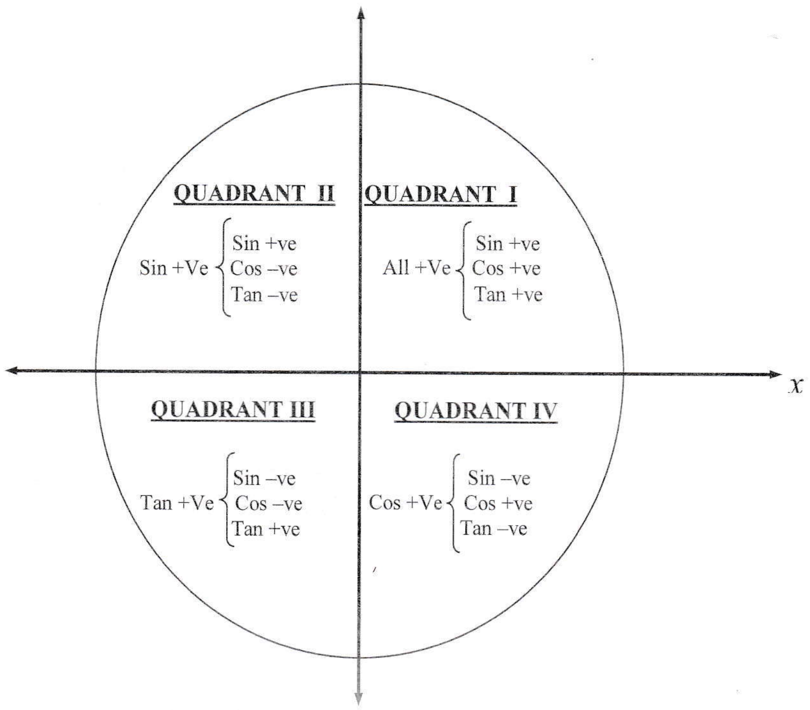


Figure 2.5

To obtain q° in the second quadrant: 180° - a° = q°
To obtain q° in the third quadrant: a°- 180° = q°
To obtain q° in the fourth quadrant: 360° - a° = q°

Note: In all quadrants discussed you have seen that the values of x and y components are $r\cos\theta$ and $r\sin\theta$ respectively. If we apply Pythagoras theorem to get the value of r in any right angled triangle across all quadrants we have:

$r^2 = x^2 + y^2$ but $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$r^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$

$$r^2 = r^2(\cos^2\theta + \sin^2\theta)$$

This becomes:

Also, from the relation that $\tan\theta = \frac{opposite}{adjacent}$ in our right angled triangle across quadrants we have:

$\tan\theta = \frac{y}{x}$, but $x = r\cos\theta$ and $y = r\sin\theta$. Then:

$$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{\sin\theta}{\cos\theta}$$

Therefore, $\tan\theta = \frac{\sin\theta}{\cos\theta}$

This shows the relationship exists in trigonometric ratios.

Example 1

Write the signs of the following trigonometric ratios:

(a) sin 170° (b) cos 240° (c) tan 310°

Solution

(a) 170° is in the second quadrant, hence sin 170° is positive

$\therefore$ sin 170° is positive.

(b) 240° is in third quadrant, hence cos 240° is Negative

$\therefore$ cos 240° is negative.

(c) 310° is in the fourth quadrant, hence tan 310° is negative

$\therefore$ tan 310° is negative.

Example 2

Express the following in terms of the sine, cosine or tangent of an acute angle:

(a) tan 95° (b) cos 165° (c) sin 317°.

Solution

(a) tan 95°.

95° is in the second quadrant

$= (180° - \theta°)$

$\theta° = 95°$

$= (180 - 95°)$

$= 85°$

$\tan 95° = -\tan(180° - 95)$

$\therefore \tan 95° = -\tan 85°$

(b) cos 165°, is in the second quadrant.

$= (180° - \theta°)$

$\theta° = 165°$

$= (180 - 165)$

$= 15°$

$\cos 165° = -\cos(180 - 165)$

$\therefore \cos 165° = -\cos 15°$

(c) sin 317° is in fourth quadrant.

$\theta° = (360° - \theta°)$

$\theta° = 317°$

$\sin 317° = -\sin(360° - 317°)$

$\therefore \sin 317° = -\sin 43°$

Example 3

Find sin q, cos q and tan q if the terminal side (hypotenuse) of θ° is the point Q (-5, -12),

Solution

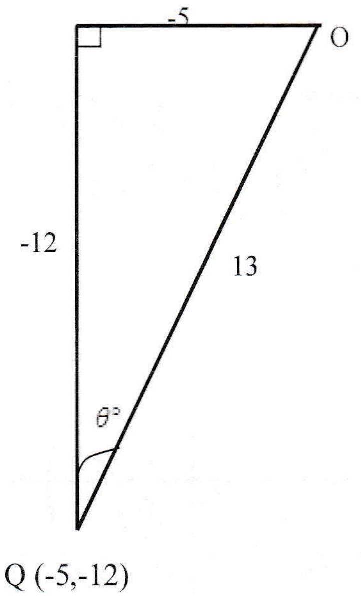


Figure 2.6

Applying Pythagoras theorem

$OQ = \sqrt{(-5)^2 + (-12)^2}$

$OQ = \sqrt{25 + 144}$

$OQ = \sqrt{169}$

$OQ = 13$

$$\therefore \sin\theta = \frac{-5}{13}$$

$$\therefore \cos\theta = \frac{-12}{13}$$

$$\therefore \tan\theta = \frac{-5}{12}$$

Example 4

Find cosq°, sinq° and tan θ° if θ° is the angle made by the positive x-axis and the line from the origin to each of the following points:

(a) (2, 6) (b) (12, -5)

Solution

(a) (2, 6)

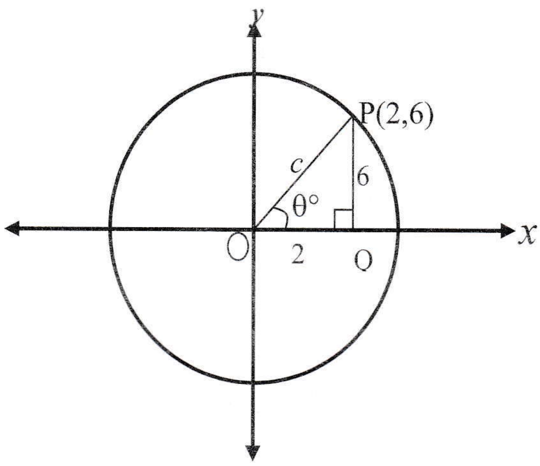


Figure 2.7

By Pythagoras theorem, from Figure 2.7

$OQ^2 + PQ^2 = OP^2$

$2^2 + 6^2 = c^2$

$C^2 = 4 + 36$

$$c^2 = 40$$

$$c = \sqrt{40}$$

$$c = \sqrt{2x2x2x5}$$

$$c = 2\sqrt{10}$$

(i) $$\cos \theta° = \frac{2}{2\sqrt{10}}$$

Rationalizing the denominator

$$\frac{2}{2\sqrt{10}} \text{ x } \frac{2\sqrt{10}}{2\sqrt{10}} = \frac{4\sqrt{10}}{4 \text{ x} 10} = \frac{\sqrt{10}}{10}$$

$$\therefore \cos \theta° = \frac{\sqrt{10}}{10}$$

(ii) $$\sin \theta° = \frac{6}{2\sqrt{10}}$$

Rationalize denominator

$$\frac{6}{2\sqrt{10}} \text{ x } \frac{2\sqrt{10}}{2\sqrt{10}} = \frac{12\sqrt{10}}{4 \text{ x } 10} = \frac{3\sqrt{10}}{10}$$

$$\therefore \sin \theta° = \frac{3\sqrt{10}}{10}$$

(iii) $$\tan \theta° = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{6}{2} = 3$$

$$\therefore \tan \theta° = 3.$$

$$c^2 = 40$$

$$c = \sqrt{40}$$

$$c = \sqrt{2x2x2x5}$$

$$c = 2\sqrt{10}$$

(i) $\cos\theta° = \dfrac{2}{2\sqrt{10}}$

Rationalizing the denominator

$$\frac{2}{2\sqrt{10}} \text{ x } \frac{2\sqrt{10}}{2\sqrt{10}} = \frac{4\sqrt{10}}{4 \text{ x} 10} = \frac{\sqrt{10}}{10}$$

$$\therefore \cos\ \theta° = \frac{\sqrt{10}}{10}$$

(ii) $\sin\theta° = \dfrac{6}{2\sqrt{10}}$

Rationalize denominator

$$\frac{6}{2\sqrt{10}} \text{ x } \frac{2\sqrt{10}}{2\sqrt{10}} = \frac{12\sqrt{10}}{4 \text{ x } 10} = \frac{3\sqrt{10}}{10}$$

$$\therefore \sin\ \theta° = \frac{3\sqrt{10}}{10}$$

(iii) $\tan\theta° = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{6}{2} = 3$

$$\therefore \tan\ \theta° = 3.$$

(b) (12, -5)

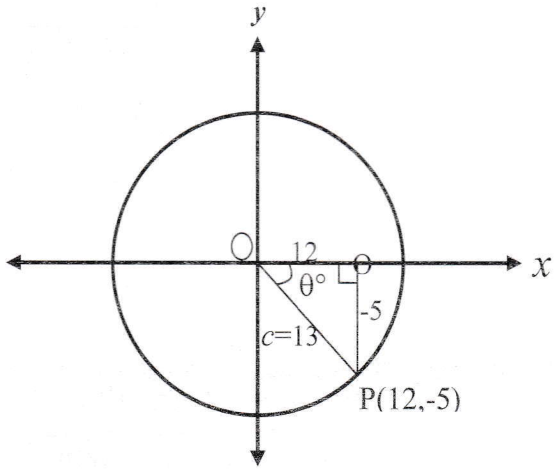


Figure 2.8

By Pythagoras theorem, from Figure 2.8

$a^2 + b^2 = c^2$

$12^2 + (-5)^2 = c2$

$c^2 = 144 + 25$

$c^2 = 169$

$c = \sqrt{169}$

$c = 13.$

$$\cos \theta° = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{OQ}{OP} = \frac{12}{13}$$

$$\therefore \cos \theta = \frac{12}{13}$$

$$\sin \theta° = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{PQ}{OP} = \frac{-5}{12}$$

$$\therefore \sin \theta° = \frac{-5}{12}$$

$$\tan \theta^\circ = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{PQ}{OQ} = \frac{-5}{12}$$

$$\therefore \tan \theta^\circ = \frac{-5}{12}$$

Activity 1

1. Write the signs of each of the following:
   (a) $\cos 160^\circ$
   (b) $\sin 310^\circ$
   (c) $\tan 75^\circ$
   (d) $\sin 220^\circ$
   (e) $\cos 355^\circ$
   (f) $\tan 190^\circ$
2. Find the values of $\cos q^\circ$, $\sin q^\circ$ and $\tan q^\circ$ if q° is an acute angle made by the x-axis and line from the origin to each of the following points.
   (a) (2, 6)
   (b) (-12, -5)
   (c) (-4, -3)
   (d) (3, 4)
3. Express the following in terms of the sine, cosine and tangent of acute angle.
   (a) $\sin 333^\circ$
   (b) $\tan 268^\circ$
   (c) $\cos 103^\circ$

Compare your answers with the answers given at the end of this unit.

Positive and Negative Angles

Dear learner, in the previous section you have learned signs of all trigonometric ratios in all quadrants. In this section you are going to learn about positive and negative angles.

Angles may be positive or negative depending on the direction in which the angle is measured. An angle may be measured in a clockwise or anti-clockwise direction. Angles measured in the clockwise direction from the positive x-axis are **negative.** Angles measured in the anti-clockwise direction from positive x-axis are **positive.**

Therefore, in this section you are going to study the angles measured in the clockwise and anti-clockwise direction. At the end of this section you will be able to identify the angles which are measured in the clockwise and anti-clockwise directions. And finally you will be able to apply the knowledge of the positive and negative angles in solving various problems related to it.

Consider the x-y plane which shows the positive and negative angles as shown in figure 2.9.

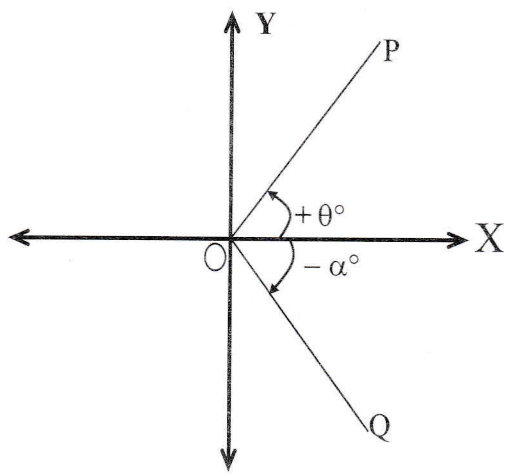


Figure 2.9

From the figure 2.9, you will see that angle $X\hat{}P$ is a **positive angle** because it is measured from the anti clockwise direction from the positive axis. Angle $X\hat{}Q$ is a **negative angle** because it is measured in clockwise direction from the positive x-axis. You can easily remember the clockwise and anti-clockwise directions by using the symbols of N for anticlockwise and S for clockwise direction.

Thus:

(a)

Angles in Anti-clockwise direction are **positive.**

(b) Angles in clockwise direction are **negative.**

Also positive and negative angles can be located in the four quadrants with the Corresponding trigonometric ratios as shown in figure 2.10 and figure 2.11.

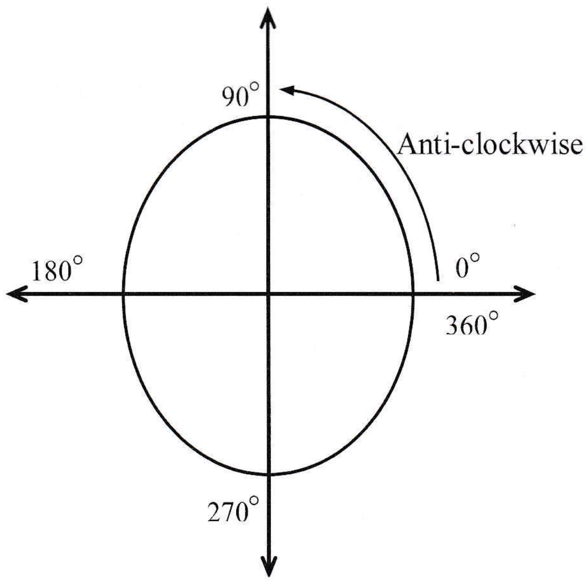


Figure 2.10

From Figure 2.10, you will see that the angle q° is measured in anti-clockwise direction.

Then, if θ° is positive, the negative angle,a° corresponding to q° is given by (-360° + θ°).

Thus, $\alpha° = 360^0 + \theta°$

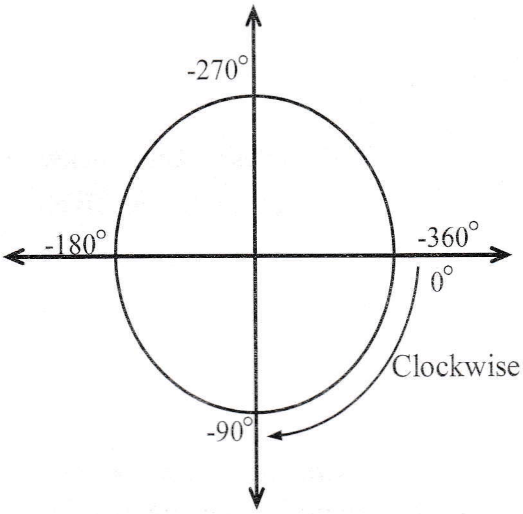


Figure 2.11

From Figure 2.11, you can see that the angle θ° is measured in clockwise direction.

Then, if θ° is negative, the positive angle, α°, corresponding to q° is given by (360° + θ°).

$\alpha° = 360^0 + \theta°$

Examples

1. Find positive or negative angles corresponding to each of the following angles:
   (a) 273°
   (b) -210°
   (c) 304°
   (d) -115°

Solution

(a) 273°

$\theta°$ - Positive

$\alpha° = -360° + \theta°$

$\alpha° = -360° + 273°$

$\therefore \alpha° = -87°$

(b) -210°

$\theta°$ - Negative

$\alpha° = 360 + \theta°$

$\alpha° = 360 - 210°$

$\alpha° = 150°$

$\alpha° = 150°$

(c) 304°

$\theta°$ - Positive

$\alpha° = -360° + \theta°$

$\alpha° = -360° + 304°$

$\therefore \alpha° = -56$

(d) - 115°

$\theta°$ - Negative

$\alpha° = 360° + \theta°$

$\alpha° = 360° - 115°$

$\therefore \alpha° = 245°$

2. Find the positive and negative angles corresponding to each of the following angles:

(a) 144°

(b) -231°

(c) -70 °

(d) 310°

Solution

(a) $144°$

$\theta°$ - Positive

$\alpha° = -360° + \theta°$

$\alpha° = -360° + 144°$

$\therefore \alpha° = -216°$

(b) $-231°$

$\theta°$ - **negative**

$\alpha° = 360° + \theta°$

$\alpha° = 360° - 231°$

$\therefore \alpha° = 129°$

(c) $-70°$

$\theta°$ - Negative

$\alpha° = 360° + \theta°$

$\alpha° = 360° - 70$

$\alpha° = 290°$

(d) 310

$\theta°$ - Positive

$\alpha° = -360° + \theta°$

$\alpha° = -360° + 310°$

$\therefore \alpha° = -50°$

Activity 2

1. Find the positive or negative angles corresponding to each of the following angles.

   (a) 186°

   (b) -110°

   (c) 302°

   (d) -225°

2. Find the positive or negative angles corresponding to each of the following:

   (a)-216°

   (b) 208°

   (c)-312°

   (d) 150°

3. Find the angle which is corresponding to 160°.
4. Find the angle which is corresponding to -89°
5. Find the positive or negative angles corresponding to each of the following:

   (a) -30°

   (b) 30°

Compare your answers with the answers given at the end of this unit.

Applying Trigonometric Ratio in Daily Life

In previous section you learnt how to identify negative and positive angles. In this section you will apply the trigonometric ratio in solving the problems which are rated to the real life of human being.

The trigonometric ratios can be used in solving practical problems with the application of angle of elevation, angle of depression and bearing.

Example 1.

A man is 172 cm tall, and the length of his shadow is 158cm. Find the angle of elevation of the sun.

Solution

You have to sketch a figure according to the instructions given. See Figure 2.12.

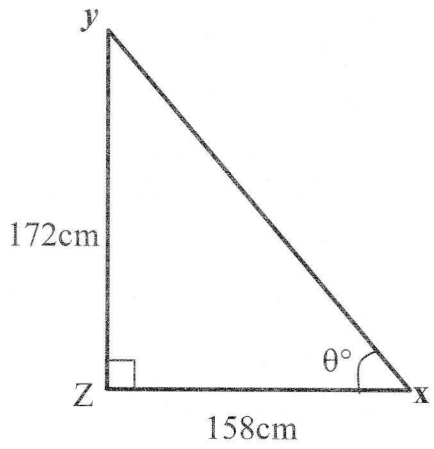


Figure 2.12

By applying the tangent of θ°, the angle of elevation of the sun is given by

$$\tan \theta° = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{172\text{cm}}{158\text{cm}}$$

$$\tan \theta° = \frac{172\text{cm}}{158\text{cm}} = 1.089$$

$$\tan \theta° = 1.089$$

By taking the inverse of tangent 1.089 you will have:

$\tan^{-1} 1.089 = \theta°$

$\theta° = \tan^{-1} 1.089$ (from mathematical tables of tangent)

$\theta° = 47° 26'$

∴ The angle of elevation of the sun is 47° 26'.

Example 2

Petro starts from point P and cycles 19.8 km to the point of a direction N41°22'W, how far has he travelled West and North respectively?

Solution

Let x and y be the distance in km due West and North of P respectively as shown in Figure 2.13

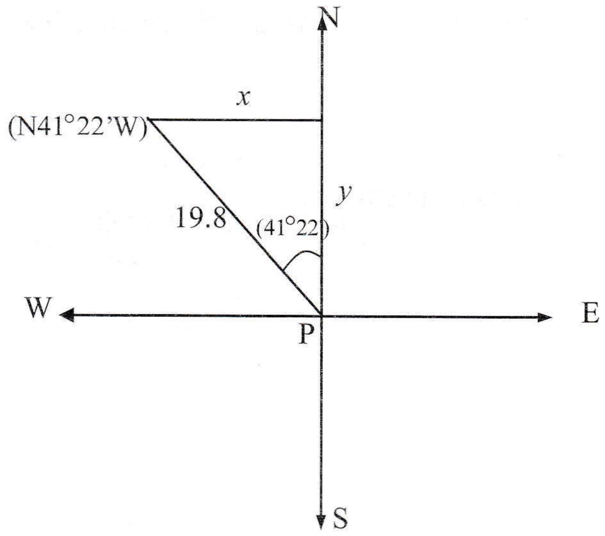


Figure 2.13

$$\text{Sin } 41°22' = \frac{x}{19.8} \Rightarrow x = 19.8 \text{ x } \sin 41°22'$$

But sin 41°22' = 0.6608 (from mathematical tables of sines)

$\therefore x = 19.8 \text{ x } 0.6608$

$\therefore x = 13.08\text{km}$

$$\text{Cos } 41° 22' = \frac{y}{19.8}$$

$\Rightarrow y = 19.8 \text{ x } \cos 41°22'$

But cos 41° 22' = 0.7505 (from mathematical tables of cosines)

$y = 19.8 \times 0.7505$

$y = 14.86\text{km}$

$\therefore$ Petro travelled 13.08km West of P and 14.86 km North of P

Example 3

From a certain point A, Fatuma observes the angle of elevation of the top of a tower to be 32°. Moving 30 m further away to a point B on the same horizontal level as the bottom of the tower C, she observes the angle of elevation to be 22°. Find the distance AC and the height of the tower.

Solution

Let h be the height of the tower and x be the distance AC as shown in figure 2.14

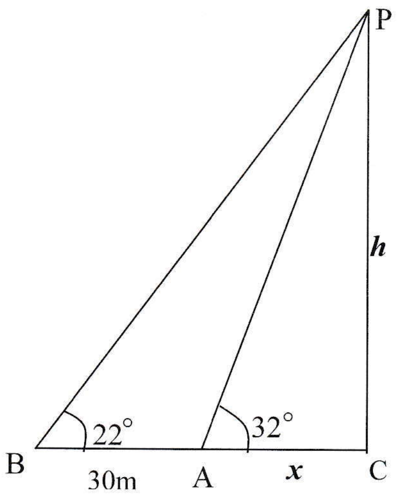


Figure 2.14

From the figure 2.14, $\tan 32° = \frac{h}{x}$

$\Rightarrow h = x \tan 32°$.................... (i)

$\tan 22° = \frac{h}{x+30}$

$\Rightarrow h = (x+30)\tan 22°$,

$h = x \tan 22° + 30 \tan 22°$............(ii)

Combine equation (i) and (ii)

$x \tan 32° = x \tan 22° + 30 \tan 22°$

$x \tan 32° - x \tan 22° = 30 \tan 22°$

$x (\tan 32° - \tan 22°) = 30 \tan 22°$

x = 30 tan 22tan 32tan 22

But tan 32° = 0.6249 and tan 22° = 0. 4040 (from mathematical tables)

$$x = \frac{30 \text{ x } 0.4040}{0.6249 - 0.4040}$$

$$x = \frac{12.12}{0.2209} = 54.9$$

$$x = 54.9\text{m}$$

∴ Distance of AC = 54.9m

Since $h = x \tan 32°$

$h = 54.9 \text{ x } 0.6249$

$h = 34.3$ m

∴ The height of the tower is 34.3 m.

Example 4:

A woman sees a tree on the other side of a river, on a bearing of 30^0. After walking 100m along the bank, the tree is directly opposite on a bearing of 45^0. How wide is the river?

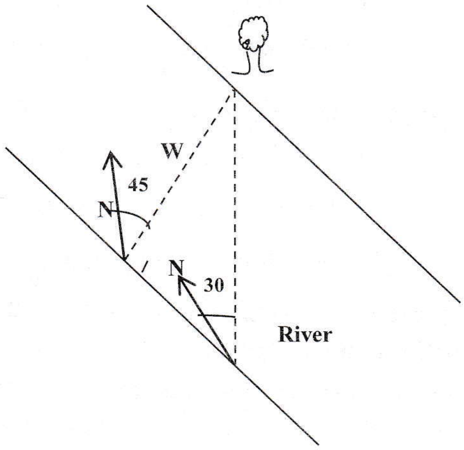


Figure 2.15

Solution

The diagram shows the situation.

Let the width of the river be w (m). the difference between the bearings is the difference between the two directions. Hence

$$\tan 15^0 = \frac{100}{w}$$

$w = 100 \div \tan 15^0$

The width of the river is 373.2 m.

Note: the width is also given by $w = 100 \times \tan 75^0$.

Activity 3

1. The edges of rectangular garden are 163 m long and 109 m wide. Find the angles made by a canal and the edge if the canal cuts the garden diagonally.
2. A ship starts a point P and travels 22 km in a direction N 32^0 41´E How far north and east of P is the ship?
3. The angle of depression of a boat from the cliff 25m is 12^0.

Find the distance of the boat from the bottom of the cliff.

4. At a point 182m from the foot of a tower on a level road, the angle of elevation of the top of the tower is 360 44′. Find the height of the tower.

Compare your answers with the answers given at the end of this unit.

Unit Reflection

Dear learner, you have completed unit two. Can you apply the sine, cosine and tangent of an angle measured in clockwise and anticlockwise directions to solve related problem in your daily life?

Can you apply trigonometric ratios in finding height of the tower, width, level, height of trees, rivers mountains, buildings and the like?

Unit Assignment

1. Write down the values of the following:
   (a) cos 110^0
   (b) sin 308^0
   (c) tan 940
   (d) cos(-2730)
   (e) sin (-4000)
   (f) sin (1,4000)

2. A tree is 5 m high. The point x is on ground 20 m from the base of the tree. What is the angle of depression of x from the top of the tree?

3. The sine of an obtuse angle is $\frac{3}{4}$. Find the cosine.

4. Find x if sin x = $\frac{-4}{3}$ and cos x = $\frac{3}{5}$

5. If the terminal side of op is p(-2,-5) , find:

(i) Sin θ°
(ii) Cos θ°
(iii) Tan θ° , where θ° is an angle which *op* makes with the *x-axis*

6. Suppose tan x = 0.7. Find x, and hence find two possible values of cos x and sin x.

7. From the top of a cliff 35 m high the angles of depression of two boats lying in a line due east of the cliff are 27° and 23°. Find the distance between the boats.

8. For the following journeys, find the distance North or South and the distance East or West:
   (a) 860 km on a bearing of 037^0
   (b) 1230 km on a bearing of 140^0
   (c) 80 km on a bearing of 250^0
   (d) 180 km on a bearing of 325^0

9. A plane flies for 200 km roughly North – East, ending 130 km North of its starting point. What is the bearing on which it has flown.

10. Show that P (0.6, 0.8) lies on the unit circle find the angle that OP makes with x-axis.

Activities Answers

Activity 1

1.(a) Negative
(b) Negative
(c) Positive
(d) Negative
(e) Positive
(f) Positive

2. (a) $\text{Sin } \theta = \frac{-3\sqrt{10}}{10}$, $\text{Cos } \theta = \frac{\sqrt{10}}{10}$, $\tan \theta = -3$

3. (a) $\text{Sin } \theta = \frac{-12}{13}$ $\text{Cos } \theta = \frac{-5}{13}$ $\tan \theta$ $\frac{12}{5}$

4.

(c) $\text{Sin } \theta = \frac{-1}{5}$ $\text{Cos } \theta = \frac{-3}{5}$ $\tan \theta$ $\frac{4}{3}$

(d) $\text{Sin } \theta = \frac{4}{5}$ $\text{Cos } \theta = \frac{3}{5}$ $\tan \theta$ $\frac{4}{3}$

5. (a)- Sin 27^0 , Cos 27^0, -tan 27^0

(b) - Sin 88^0 , -Cos 88^0, tan 88^0

(c) Sin 77^0 , -Cos 77^0, -tan 77^0

1. Activity 2

1. (a) -174^0

(b) 250^0

(c) -58^0

(d) 135^0

2. (a) 144^0

(b) -152^0

(c) 48^0

(d) -210^0

3. Several angles like 20^0, 200^0...

4. Several angles like $91^0, 271^0$ ...

5. (a) 330^0

(b) -330^0

Activity 3

1. $33^{0} 46'$ and 56^{o} 14'
2. 18.52 km East, 11.88 m North
3. 5.31 km
4. 135.82 km

Unit 3

Describing Sines and Cosines Functions

Introduction

In the unit 2 of this module you learnt how to get sine, cosine and tangent of an angle measured in the clockwise and anticlockwise directions.

In this unit you will study the sine and cosine, functions and the techniques of drawing their graphs. At the end of this section you will be able to draw the graphs of the trigonometric functions and also to apply the graphs in solving various problems which are related to it.

The relationship between the angles and its trigonometric ratios defines a function. For example if sin $\theta° = y$, then the ordered pair $(\theta°, y)$ defines the sine function. Similarly if cos $\theta° = x$, then the ordered pair $(\theta°, x)$, defines the cosine function and if tan $\theta° =$ yx then, the ordered pairs ($\theta°$,yx) define the tangent function. Any trigonometric function can be an even or odd function.

Learning Outcomes

Upon completion of this unit you will be able to:

- Draw and interpret the graph of sine functions; and
- Draw and interpret the graph of cosine functions

Drawing and Interpreting the Graph of Sine Functions

In this section you are going to study about the sine function as an odd function and how to draw the graph of sine function. Also you will be able to apply the graph in solving problems related to it.

The function is said to be an odd function if f of minus x is equal to minus f of x, that is if $f(-x) = -f(x)$. Therefore, $f(-x)$ in sine function is equal to $-f(x)$. Hence, the sine function is an odd function because $f(-x)$ is equal to $-f(x)$. For example:

sin (-45°) = -sin 45°
sin -135° = -sin 135°
sin -240° = -sin 240°

Graph of Sine Function

After you have familiarized yourself with positive and negative sine of angles, let us see how to draw its graphs. The graph of sine function is drawn by using the ordered pairs between -720° and 720°. This interval between 720° and -720° is called the domain of the function, and is normally written in the form given by: $-720° \leq \theta° \leq 720°$

Therefore, in order to draw the graph of the sine function, first you have the domain which are called ordered pairs $-720° \leq \theta \leq 720°$. Second you must make a table of values which shows the values of sines of each angle from the ordered pairs between -720° and 720°. After filling the table of values of the sine function, then you use the table of values to draw the graph of the function

Example

Draw the graph of sine function from the interval between -720° and 720°.

Solution

The sine function is defined as $y = \sin \theta°$

Table 3 shows the ordered pairs of the sine function for angles between -720° and 720°

Table 3.1

θ°	-720	-630	-540	-450	-360	-270	-180	-90	0	90	180	270	360	450	540	630	720
y= sin θ°	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0

These ordered pairs from Table 3.1 can be plotted to get the graph of sine function as shown in Figure 3.1.

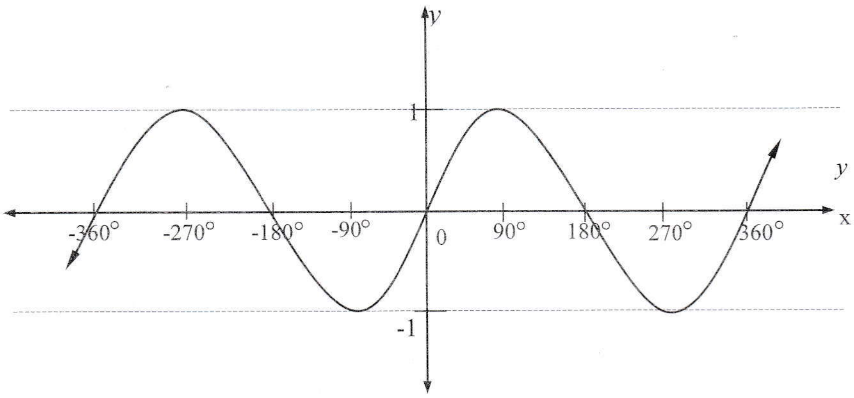


Figure 3.1

From Figure 3.1, you can see that the graph of the function of Sin q° shows that the value of the functions repeats at intervals of 360°. Since the sine function repeats at intervals of 360°, the sine function is therefore called a **periodic function** and the interval 360° is called **period** of the function.

A function that does not repeat itself at an equal interval is called non periodic function.

Thus, the value of sin -360° is the same as the value of sin 0°, and also the value of sin -270° is equal to the value of Sin 90°. The interval in between is 360°. That is the difference in interval of the sine function is given by:

$$\sin \theta^\circ = \sin (\theta^\circ + 360^\circ)$$

Drawing and interpreting graphs of cosine functions

As you have learnt sine as an odd function cosine is considered as an even function. The function is said to be even if f of negative x is equal to f of positive x, that is $f(-x) = f(x)$, therefore, the function is called the even function. Hence the cosine function is an even function because the $f(-x)$ is equal to $f(x)$.

For example

Cos (-45) = Cos 45°

Cos (-120) = Cos 120

Cos (-240°) = Cos 240

In this subsection, you will study the cosine function as an even function and the graph of the cosine function. At the end of this subsection you will be able to draw the graph of the cosine function and to apply the graph of the cosine function in solving various problems which are related to this unit as a whole.

Graph of Cosine Function

The graph of cosine function is drawn by using the ordered pairs between -720° and 720°. The interval between -720° and 720° is called the domain of the function. This domain is normally written in form given as:

$$-720° \leq \theta° \leq 720°.$$

To draw a graph of the cosine function, you have to follow three steps. The steps are, first, to get the domain for the function which is ordered pairs given by the interval -720° and 720°,The second, is to make the complete table of values. The third step, is to draw the graph by using the values given in your table of values.

Example

Draw the graph of cosine function from the interval between -720° and 720°

Solution

The cosine function is defined as y = cos θ°;

Table 3.2 shows the ordered pairs of the cosine function for angles between -720 and 720°.

Table 3.2

θ°	-720	-630	-540	-450	-360	-270	-180	-90	0	90	180	270	360	450	540	630	720
y= cos θ°	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0

The ordered pairs in Table 3.2 can be plotted to get the graph of the cosine function y = cos q° as shown in figure 3.2 below.

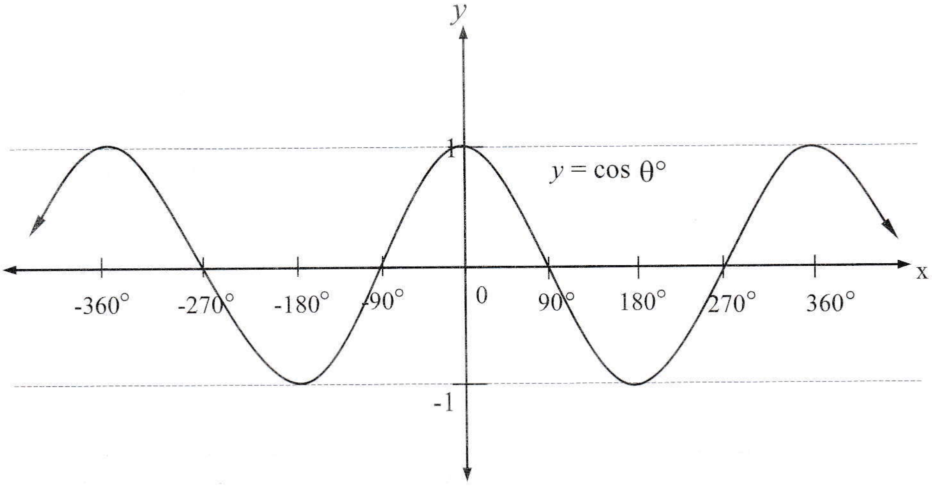


Figure 3.2

From Figure 3.2, you can see that the graph of the cosine function shows that the values of the functions repeat at the interval of 360°. That is, the value of cos -360°, is the same as the value of cos 360° and also the value of cos -180° is equal the value of cos 180°. Since the cosine function repeats at interval of 360°, the function is called **periodic function** and the interval 360° is called the **period** of the function.

Thus, the difference in interval of the cosine function is given by:

$$\cos\theta° = \cos(\theta° + 360°)$$

Example

Use the trigonometric graphs in the interval $-360 \leq \theta° \leq 360°$ to find q° such that:

(a) $\sin\theta° = 0.4$ (b) $\cos\theta° = 0.5$ (c) $\tan\theta° = -0.5$

Solution

You have to draw the graph of each the function and hence use it to find the value of θ° by looking at the corresponding given value.

(a) Sin θ° = 0.4

Draw the graph of sine function (see Figure 3.3).

From the graph you look the points where the line $y = 0.4$ cuts the graph of sine function and read the answers on the x-axis

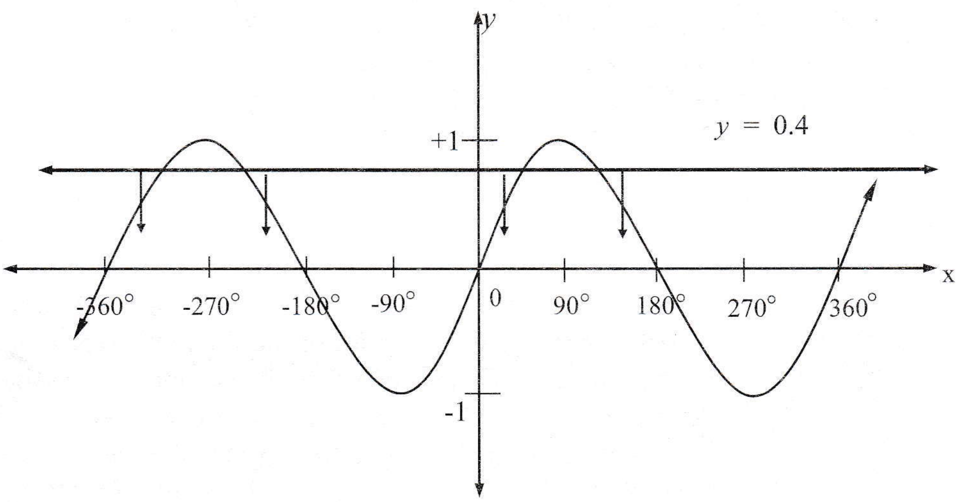


Figure 3.3

sin θ° = 0.4

$\sin^{-1} 0.4 = \theta°$

θ° = $\sin^{-1}0.4$ (from mathematical tables of sine inverse)

∴ θ° = -336°, -204°, 24°, 156°

(b) cos θ° = 0.5

You have to draw the graph of the cosine function and a line of $y = 0.5$ and check the angles which are corresponding with the intersection of the line $y = 0.5$ and the graph of cosine function as you can below, in Figure 3.4.

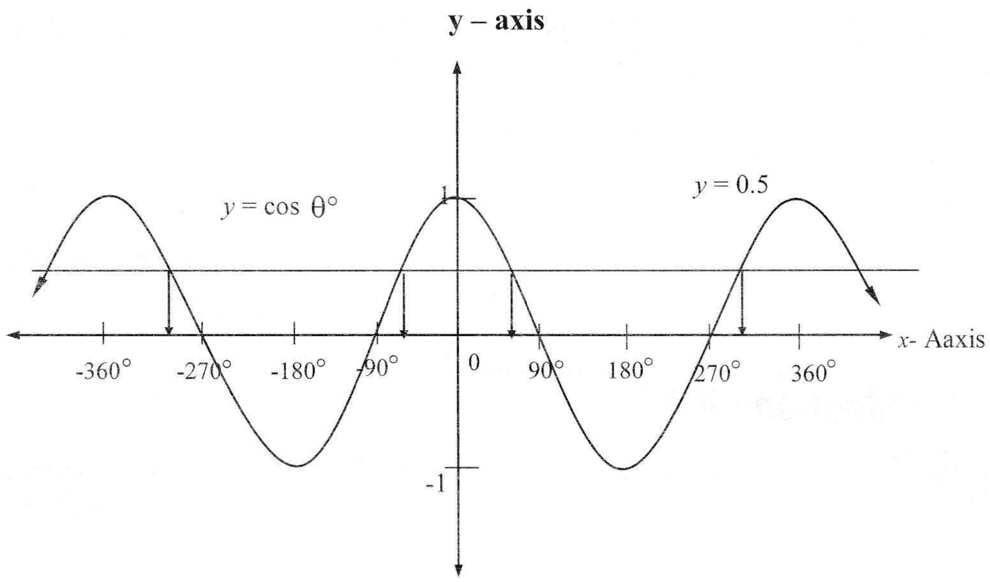


Figure 3.4

From Figure 3.4, you see that the graph of $y = 0.5$ cuts the graph of the cosine function at four points. These points are corresponding to the angles, in the x-axis,

-300°, -60°, 60°, and 300

Thus,

$\cos \theta° = 0.5$

$\text{Cos}^{-1} 0.5 = \theta°$

$\theta° = \text{Cos}^{-1} 0.5$ (from mathematical tables of cosine inverse)

$\therefore \theta° = -300°, -60°, 60°$ and $300°$

Activity 1

1. Using the trigonometrical graphs, find the sine and cosine of each of the following angles:

(a) -300° (d) 450°

(b) -40° (e) -520°

(c) 155°

2. Using the graphs of trigonometrical from interval of $-360 \leq \theta° \leq 360°$, find the value of $\theta°$ such that

   (a) $\sin \theta° = + 0.45$

   (b) $\cos \theta° = -0.5$

   Compare your answers with the answers given at the end of this unit.

Unit Reflection

You have completed this unit; can you identify areas where the knowledge of trigonometric functions can be applied? What are the areas from this unit are still difficult to you? How can you do to clear your doubts?

Unit Assignment

1. Let $y = \frac{1}{2} \sin x + \frac{1}{3} \cos x$

Table3.3

x	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	330^0	360^0	390^0	420^0	450^0
y															

a) Fill in the table 3.3.

b) Plot graph of the function.

c) From your graph solve the equation:

   (i) $\frac{1}{2} \sin x + \frac{1}{3} \cos x = 0.3.$

   (ii) $\frac{1}{2} \sin x + \frac{1}{3} \cos x = 0.2.$

d) Find the least possible value of $\frac{1}{2} \sin x + \frac{1}{3} \cos x$.

2. Fill in table 3.4 and plot the graph of y=cos x, for between 0° and 450°

Table 3.4

x	0^0	30^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	330^0	360^0	390^0	420^0	450^0
Cos x															

3. (a) Make the table and draw the graph of sin x + cos x using intervals from $-720^0 \leq \theta \leq 720^0$

   (b) Use the graph to solve

   (i) Sinx + cosx=0.6

   (ii) Sinx +cosx = 1.2

4. (a) Make the table and plot the graph of the function, using intervals from

   $-720° \leq \theta° \leq 720°$

   (b) From your graph solve the equation

   (i) 2sinx + 3cosx=1.7

   (ii) 2sinx + 3cosx=0.9

   (c) From your graph, find the greatest possible value of 2sinx + 3cosx

5. Use the graph of cosq to find the value of q if 5cosq = 1.5, for

   $0 \leq \theta \leq 360°$

6. Use the graph of y=cos θ to find the angles for which:-

   (a) Cos θ = 0.2

   (b) Cos θ = 0.55

   (c) Cos θ = 0.65

7. Find two solutions between 0^0 and 360^0.

(a) Cos $x = 0.5$
(b) Sin $x = 0.6$
(c) Tan $x = -1$

Activities Answers

Activity 1

1. (a).$\text{Sin}(-300^0) = 0.9$

   $\text{Cos}(-300^0) = 0.5$

   (b) $\text{Sin}(-40^0) = 0.6$

   $\text{Cos}(-40^0) = 0.7$

   (c) $\text{Sin } 155^0 = 0.4$

   $\text{Cos } 155^0 = -0.9$

   (d) $\text{Sin } 450^0 = 1$

   $\text{Cos } 450^0 = 0$

   (e) $\text{Sin } (-520^0) = -0.3$

   $\text{Cos } (-520^0) = -0.9$

2. (a) $\theta = 26^0 44', 153^0 16'$

   $-333^0 16', -206^0 44'$

   (b) $\theta = 120^0, 240^0$

   $-240^0, -120^0$

Unit 4

Deriving and Applying Sine and Cosine Rules and Compound Angle Formulae

Introduction

Dear learner, in the previous unit you have learnt how to draw and interpret graphs of sines and cosine functions. It is challenge to find the sides and angles of non right angled triangles but this can be overcome by use of other trigonometric functions. In this unit you will learn how to derive the sines and cosine rules which are used in solving problem concerning non-right angled triangles. Also how to apply compound angle formulae for sines, cosines and tangent in solving trigonometric problems.

Learning Outcomes

Upon completion of this unit you will be able to:

- Derive and apply sine rule;
- Derive and apply cosine rule; and
- Apply the compound angle formulae for sine, cosine and tangent in solving trigonometric problems.

Deriving and Applying Sine Rule

Dear learner, you have completed unit 3 of this module. In the unit you learnt about sines functions. In this section you are going to learn about deriving and applying sine rules which is related to trigonometric ratio as well as sine functions. The sine rule can be stated as "in any triangle the sides are proportional to the sines of the opposite angles"

Derivation of Sine Rule

Consider a triangle ABC with side marked by a, b, and c, as shown in Figure 4.1.

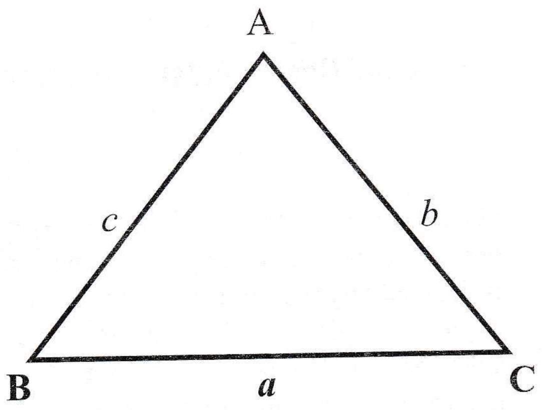


Figure 4.1

From Figure 4.1 you can see that the area of triangle ABC can be given by three formulas as:

$$\text{Area of D ABC} = \frac{1}{2}\, ab \sin C \quad \ldots\ldots\ldots\ldots\ldots \text{i}$$

$$\text{Area of D ABC} = \frac{1}{2}\, ac \sin B \quad \ldots\ldots\ldots\ldots\ldots \text{ii}$$

$$\text{Area of D ABC} = \frac{1}{2}\, bc \sin A \quad \ldots\ldots\ldots\ldots\ldots \text{iii}$$

Since the area of the triangle doesn't change, then equation (i) = equation (ii) = equation (iii)

$$\text{Thus,} \quad \frac{1}{2}\, ab \sin C = \frac{1}{2}\, ac \sin B = \frac{1}{2}\, bc \sin A$$

If you multiply each expression by $\frac{2}{\text{abc}}$ you will obtain

$$\frac{1}{\cancel{2}}\,\cancel{ab} \sin C \text{ x } \frac{\cancel{2}}{\cancel{abc}} = \frac{1}{\cancel{2}}\,ac \sin B \text{ x } \frac{2}{\cancel{abc}}$$

$$= \frac{1}{\cancel{2}}\,\cancel{bc} \sin A \text{ x } \frac{\cancel{2}}{abc}$$

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

This expression is called the sine rule, which states as follows: in any triangle the sides are proportional to the sines of the opposite angles. Therefore, the sine rule can be arranged in proper ways as:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The sine rule can be used in solving various problems which are concerned with trigonometric ratios problems.

Example 1

1. Find the unknown sides and angles in a triangle ABC, given that $a = 7.5$cm, $c = 8.6$cm and $= 80°.

Solution

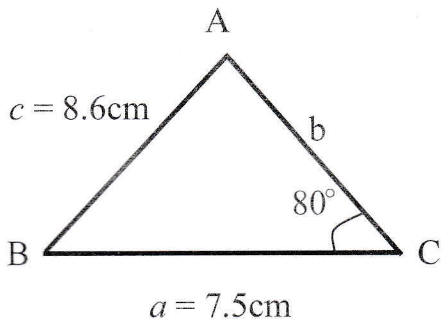


Figure 4.2

(i) **Size of angle A**

By using sine rule

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin A}{7.5} = \frac{\sin 80^0}{8.6}$$

$$\sin A = \frac{\sin 80^0 \text{ x } 7.5}{8.6}$$

but sin 80° = 0.9848 from mathematical tables

$$\sin A = \frac{0.9848 \times 7.5}{8.6}$$

$\sin A = 0.8588$

Using the mathematical table, you can get angle A by taking the inverse sine of 0.8588.

$\sin A = 0.8588$

$\sin^{-1} 0.8588 = A$

$A = \sin^{-1} 0.8588$

$A = 59° 19'$.

$\therefore$ The Angle A is 59° 19'

(ii) **Size of angle B**

Since the angle sum of any triangle is 180°,

Therefore,

$+ 59° 19' + 80° = 180°$

$= 180° - (59° 19' + 80°)$.

$\hat{B} = (180° - 130° 19')$

$\hat{B} = 40° 41'$

$\therefore$ Angle B = 40° 41'

(iii) **Side *b***

By using the sine rule, you can see that

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$b = \frac{c \sin B}{\sin C}$$

but $\hat{B} = 40° 41'$

$\hat{C} = 80°$

$c = 8.6$

$$b = \frac{8.6 \text{ x } \sin 40^0 41'}{\sin 80^0}$$

$$b = \frac{8.6 \text{ x } 0.6519}{0.9948}$$

$\therefore b = 5.7\text{cm}$

Example 2

Find the unknown sides and angles from the following

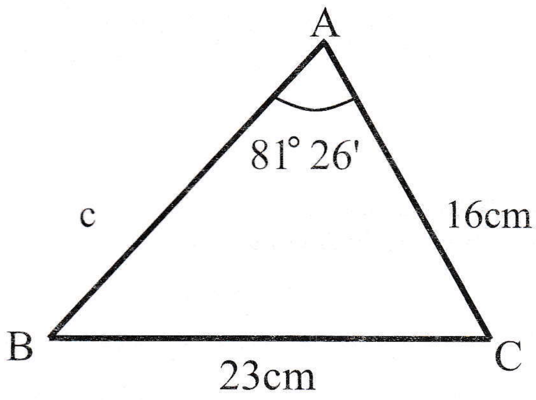


Figure 4.3

Solution

(i) Angle $\hat{B}$

From figure 4.3, by sine rule you will have

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$a = 23$

$b = 16$

$\hat{A} = 81°26'$

Sin 8126'23 = sin B16

but sin 81°26' = 0.9884

$$\sin \hat{B} = \frac{16 \times 0.9884}{23}$$

$\sin \hat{B} = 0.6876$

If you take the inverse sine of 0.6876 from the table, you will have:

$\sin \hat{B} = 0.6876$

$\sin^{-1} 0.6876 = \hat{B}$

$\hat{B} = \sin^{-1} 0.6876$

$\therefore \hat{B} = 43° 26'$

(ii) Angle $\hat{C}$

You can find angle $\hat{C}$ by using the sum of angles of any triangle which is 180° thus,

$\hat{C} + 43°26' + 81°26' = 180°$

$\hat{C} + 124° 52' = 180°$

$\hat{C} = 180° - 124° 52'$

$\hat{C} = 55°08'$

(iii) Side c:

Side c can be found using a sine rule

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$\hat{A} = 81°26'$

$a = 23$

$\hat{C} = 55°08'$

$$\therefore c = \frac{a \times \sin C}{\sin A}$$

$$c = \frac{23 \times \sin 55^0 8'}{\sin 81^0 26'}$$

$$c = \frac{23 \times 0.8205}{0.9894}$$

$$\therefore c = 19.09 \text{ cm}$$

Therefore, the unknowns are: side c = 19.09 cm, and

Angles $\hat{B} = 43° 26'$ and $\hat{C} = 55°08'$

Activity 1

1. Find the unknown sides and angles in triangle ABC, given that a = 7cm, c = 8.8cm and angle $\hat{C} = 78°$.
2. Find the unknown sides and angles in triangle ABC, in which a = 19cm, b= 30cm and angle $\hat{B} = 110°13'$.
3. Find the unknown sides and angles in each of the following triangles:

(a)

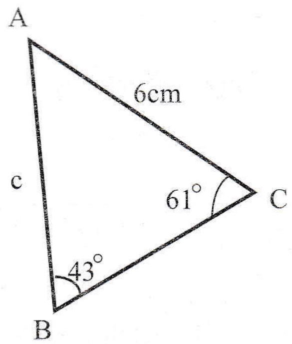


Figure 4.4

(b)

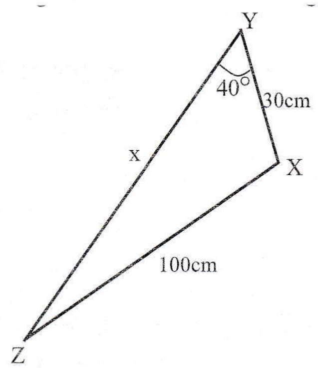


Figure 4.5

Compare your answers with the answers given at the end of this unit

Deriving and Applying Cosine Rules

Dear learner, you learnt about deriving and applying the sine rules. In this section you will learn how to derive and apply cosine rule which is also related to trigonometric ratio as well as cosine functions. This formula is applicable on problems involving two sides and one angle in between.

Derivation of Cosine Rule

Consider a triangle ABC drawn on a co-ordinate plane of x and y with vertex A at the origin, vertex B at $(c, 0)$ and vertex C at $(b\cos\hat{A}, b\sin\hat{A})$ as shown in Figure 4.6.

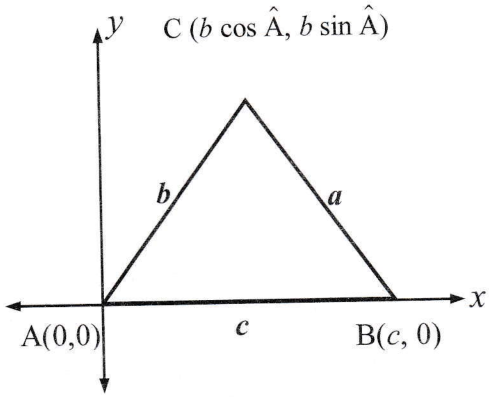


Figure 4.6

In figure 4.6, the distance BC can be written by using the distance formula.

$$BC = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

The co-ordinate of B is $(c, 0)$ and that of C is $(b\cos\hat{A}, b\sin\hat{A})$, then, the distance will be given as:

$$BC = \sqrt{(b\cos\hat{A} - c)^2 + (b\sin\hat{A} - 0)^2}$$

$$BC = \sqrt{b^2 \cos^2 \hat{A} + c^2 - 2b \cos \hat{A} + b^2 \sin^2 \hat{A}}$$

but $BC = a$

$$a = \sqrt{b^2 \cos^2 \hat{A} + c^2 - 2b \cos \hat{A} + b^2 \sin^2 \hat{A}}$$

If you square both sides you will have:

$a^2 = b^2\cos^2\hat{A} + c^2 - 2bc \cos \hat{A} + b^2 \text{Sin } \hat{A}$.

$a^2 = b^2\cos^2\hat{A} + b^2\sin^2\hat{A} + c^2 - 2bc \text{ Cos } \hat{A}$

$a^2 = b^2(\cos^2\hat{A} + \sin^2 \hat{A}) + c^2 - 2bc \text{ Cos } \hat{A}$

but $\cos^2A + \sin^2A = 1$

$a2 = b2 + c2 - \mathbf{2bc\ cos}\ \hat{A}$ This is the cosine rule

Also, if you fix the remaining vertices at the origin in turn, you will obtain the following equation of the cosine rule as:

$b2 = a2 + c2 - \mathbf{2ac} \textbf{ cos B}$
$c2 = a2 + b2 - 2ab \textbf{ cos C}$

Example

Find the unknown side and angles in a triangle ABC given that $a = 3$ cm, $c = 4$ cm and B = 30°.

Solution

Draw your triangle as in Figure 4.7.

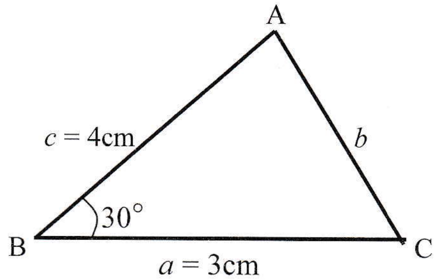


Figure 4.7

(i) **side *b***

By using the cosine rule you will see that

$b^2 = a^2 + c^2 - 2ac \cos B$

$a = 3$

$c = 4$

$b^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \cos 30°$

but $\cos 30° = 0.866$(From mathematical tables)

$b^2 = 9 + 16 - 24 \times 0.866$

$b^2 = 25 - 20.8$

$b^2 = 4.2$

$b = \sqrt{4.2}$

$\therefore b = 2.05\text{cm}$

(ii) **Angle A**

To find angles A, you use the cosine rule again as follows:

$a^2 = b^2 + c^2 - 2bc \cos A$

$a = 3$

$b = 2.05$

$c = 4$

$2bc \cos A = b^2 + c^2 - a^2$ dividing by $2bc$ both sides,

$$\frac{2b \cos \hat{A}}{2b} = \frac{b^2 + c^2 - a^2}{2b}$$

$$Cos\,\hat{A} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$Cos\,\hat{A} = \frac{(2.05)^2 + 4^2 - 3^2}{2 \times 2.05 \times 4}$$

$$Cos\,\hat{A} = \frac{4.2 + 16 - 9}{2 \times 2.05 \times 4}$$

$$Cos\,\hat{A} = \frac{11.2}{16.4} = 0.6829$$

$Cos\,\hat{A} = 0.6829 \Rightarrow A = \cos^{-1} 0.6829$ (from Mathematical tables)

$\therefore A = 46' 55'$

(iii) Angles C

You can find angle $ by using the fact that the sum of angles in a triangle is 180°

$$C + 46°55' + 30° = 180°$$

$$C = 180° - (46°55' + 30°)$$

$$C = 180° - 76°55'$$

$$C = 103°05'$$

∴ *The unknowns are:*

b = 2.05cm and angles A = 46°55' and C = 103° 05'.

Activity 2

1. Find the unknown angles in triangle ABC given $a = 5\text{cm}, b = 6\text{cm}$ *and* $c = 4\text{cm}$.
2. In a triangle ABC, a = 13cm, b= 16cm and c = 12cm. Find the middle-sized angle.
3. Find the unknown side and angles in each of the following triangles:

(a)

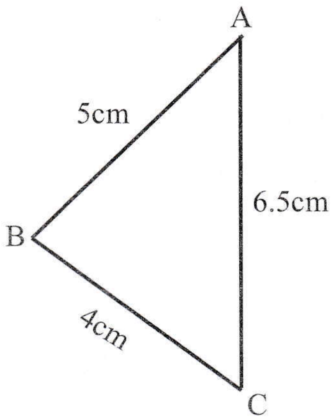


Figure 4.8

(b)

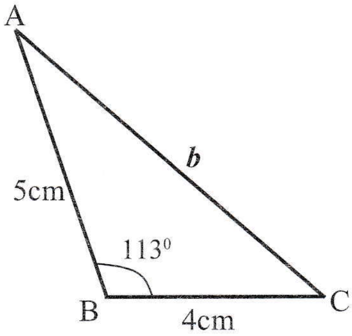


Figure 4.9

(c)

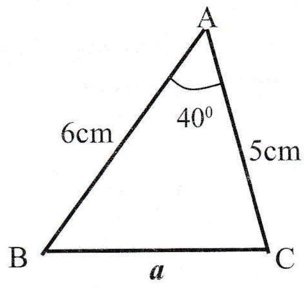


Figure 4.10

Compare your answers with the answers given at the end of this unit.

Applying Compound Angle Formulae for Sine, Cosine and Tangents

Dear learner, in previous section you have learnt about sine and cosine rules. In this section you are going to learn about other useful trigonometric relations, which can be determined by considering the sine and cosine of the sum or difference of any two angles in solving trigonometric problems. The sum or difference of any two angle in sine, cosine and tangents are called **Compound Angle Formulae**.

Though in this section you are required to learn the application of these formulae, it is very important to learn how to derive them.

Let us investigate whether $\sin(30° + 60°) = \sin 30° + \sin 60°$

$\text{Sin}(30° + 60°) = \sin 90° = 1$, $\sin 30° = ½$ and $\sin 60° = \frac{\sqrt{3}}{2} = 0.866$

Therefore,

$\sin(30° + 60°) \neq \sin 30^0 + \sin 60^\circ$ since $1 \neq 1.366$

Similarly, we can investigate for example,

Whether $\cos(180° - 90°) = \cos 180° - \cos 90°$.

$\text{Cos}(180° - 90°) = \cos 90° = 0$; $\cos 180° = -1$ and $\cos 90° = 0$.

Therefore:

cos (180° - 90°) ≠ cos 180^0 - cos 90° since $0 \neq -1$.

From these examples, it can be deducted that:

$$\sin(A+B) \neq \sin A + \sin B$$
$$\sin(A-B) \neq \sin A - \sin B$$
$$\cos(A+B) \neq \cos A + \cos B$$
$$\cos(A-B) \neq \cos A - \cos B$$

Cosine of the sum and different of any two angles

You can use the knowledge of co-ordinate geometry and cosine rule which you learnt in this module to establish the relations between the cosine of the sum and difference of any two angles or radians.

Consider unit circle with points P and Q and angles A and B as shown in Figure 4.11 Let the distance from P to Q be d and distances OP and OQ be a and b respectively where a and b are radii of a unit circle.

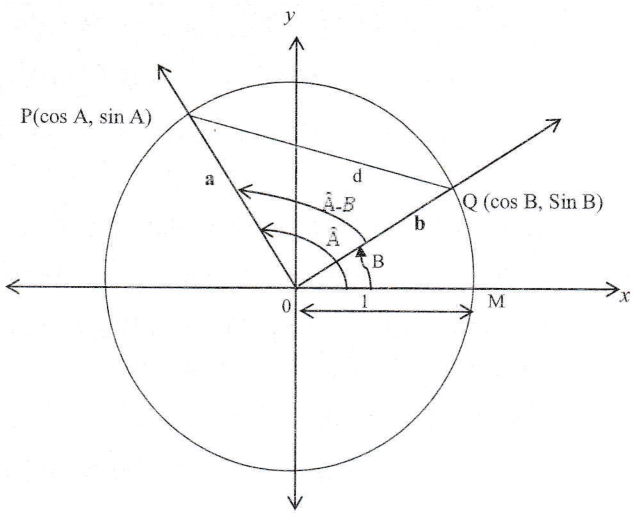


Figure 4.11

Then, by distance formula;

$d^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$

$= \cos^2 A - 2 \cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B$

$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$

$d^2 = 2 - 2 (\cos A \cos B + \sin A \sin B)$.....................................(i)

Similarly, by cosine rule,

$d^2 = a^2 + b^2 - 2ab \cos(A - B)$ but $a = 1$ and $b = 1$ (unit radii)

Therefore,

$d^2 = 1 + 1 - 2 \times 1 \times 1 \cos (A-B)$

$d^2 = 2-2 \cos (A-B)$...(ii)

Equating the expressions for d^2 obtained from the distance formula and from the law of cosines, i.e equation (i) and (ii), we get:

$2-2\cos (A-B) = 2-2(\cos A \cos B + \sin A \sin B)$

$-2\cos (A-B) = -2(\cos A \cos B + \sin A \sin B)$

$\therefore \cos (A-B) = \cos A \cos B + \sin A \sin B.$

This relation is true for all angles $\hat{A}$ and $\hat{B}$

Let us now derive the relation for cos (A+B).

From the fact the cos (-B) = cos B and sin (-B) = -sin B, cos (A + B) can be expressed as follows:

$\cos (A + B) = \cos (A-(-B)$

$= \cos A \cos (-B) + \sin A \sin (-B)$

$= \cos A \cos B + \sin A(-\sin B)$

$\therefore \cos (A+B) = \cos A \cos B - \sin A \sin B$

This relation is true for all angles $\hat{A}$ and $\hat{B}$

The relations for cos (A-B) and cos (A+B) for angles A and B will be the same for functions of numbers.

Therefore,

$$\text{Cos}\,(s-t) = \cos s \cos t + \sin s \sin t$$

$$\therefore \cos(s+t) = \cos s \cos t - \sin s \sin t$$

Example 1:

Find cos 135° from cos (90° + 45°)

Solution:

$$\text{Cos } 135° = \cos(90° + 45°)$$

$$= \cos 90° \cos 45° - \sin 90° \sin 45°$$

$$= (0 \times \frac{\sqrt{2}}{2}) - (1 \times \frac{\sqrt{2}}{2})$$

$$= -\frac{\sqrt{2}}{2}$$

Example 2:

Find the value of $\cos(\frac{\pi}{3} - \frac{\pi}{4})$

Solution:

$$\cos(\frac{\pi}{3} - \frac{\pi}{4}) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= (\frac{1}{2} \times \frac{\sqrt{2}}{2}) + (\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

Activity 3:

1. Find the value of cos 15° by using cos(45° - 30°)
2. Find cos 105° from cos (45° +60°)
3. Verify that cos (π - $\frac{2\pi}{3}$ cos π cos $\frac{2\pi}{3}$ + sin π sin $\frac{2\pi}{3}$.
4. Verify that the following relations are true for all value of t

   (a) *cos* $(\pi - t) = -\cos t$

   (b) *cos* $(2\pi - t) = \cos t$

   (c) *cos* $(\frac{\pi}{2} - t) = \sin t$

   (d) *cos* $(\frac{3}{2}\pi - t) = \sin t$
5. Verify that cos (30° + 60°) = cos 30° cos 60° - sin 30° sin 60°.
6. Verify by the formula for cos (t + s) that cos (t + 2π) = cos t.

   What does this show about the period of the cosine function?

Sine of the Sum and Difference of any two Angles

You have managed to derive compound angles formulae for cosines. Now you can use the knowledge of cosines of the sum and difference of any two angles to derive compound angles of the sum and difference of any two angles.

Consider C to be any acute angle.

Then cos (90°-C) = cos 90° cos C + sin 90° sin C

= 0 x cos C + 1 x sin C

= sin C.

Cos (90° -C) = sin C

Let C be (90° - A)

Then 90° - C = 90° - (90° - A) = A

This means that

cos (90° - C) sin C becomes cos A = sin (90° - A).

Now, let C be another name for (A+B). The formula for the sine of the sum of the two angles A and B can be found as follows:

Sin (A+B) = cos (90° - (A+B))

Sin (A+B) = cos [(90° - A) – B]

Sin (A+B)=cos(90°-A) cosB + sin(90°-A) sinB

Sin (A + B) = sin A cos B+ cos A sin B

The formula for the sine of the difference of any two angles A and *B* can also be found by considering sin [A + (-*B*)] in the formula for the sum above.

Noting that,

cos (-B) = cos B and sin (-B) = - sin B, we have

Sin [A + (-B)] = sin A cos (-B) + cos A sin (-B)

Sin (A – B) = sin A cos B + cos A (-sin B)

Sin (A – B) = sin A cos B – cos A sin B

The relations for Sin (A + B) and sin (A-B) for angles A and B will also be the same for functions of numbers.

Therefore,

Sin(s + t) = sin s cos t + cos s sin t

Sin (s+ t) = sin s cos t – cos s sin t

Example 1:

Verify that sin (90° + 30°) = sin 90° cos 30° + cos 90° sin 30°.

Solution:

LHS: $\sin(90°+30°) = \sin 120° = \sin(180° - 120°) = \sin 60° = \sqrt{\frac{3}{2}}$

RHS: $\sin 90° \cos 30° + \cos 90° \sin 30° = (1 \text{x} \sqrt{\frac{3}{2}}) + (0 \text{ x} \frac{1}{2}) = \sqrt{\frac{3}{2}}$.

$\therefore$ Cos (90° + 30°) = sin90°cos30° + cos90° sin30°

Example 2:

$$\text{Verify that } \sin\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$$

Solution:

$$\text{LHS: } \sin\left(\frac{2\pi}{3}-\frac{\pi}{6}\right)=\sin\frac{3\pi}{6}=\sin\frac{\pi}{2}=1$$

$$\text{RHS: } \sin\frac{2}{3}\pi\cos\frac{\pi}{6}-\cos\frac{2}{3}\pi\sin\frac{\pi}{6}=\left(\sqrt{\frac{3}{2}}\times\sqrt{\frac{3}{2}}\right)+\left(\frac{1}{2}\times\frac{1}{2}\right)$$

$$=\frac{3}{4}+\frac{1}{4}=1$$

$$\therefore \sin\left(\frac{2}{3}\pi-\frac{\pi}{6}\right)=\sin\frac{2}{3}\pi\cos\frac{\pi}{6}-\cos\frac{2}{3}\pi\sin\frac{\pi}{6}$$

Activity 4

1. Verify that $\sin(30° + 60°) = \sin 30° \cos 60° + \cos 30° \sin 60°$.
2. Verify that $\sin\left(\frac{2}{3}\pi + \frac{5}{3}\pi\right) = \sin\frac{2}{3}\pi \cos\frac{5}{3}\pi \sin\frac{5}{3}\pi$.
3. Use sin [s+(-t) to help find a formula for sin (s-t).
4. By using the formula for sin(A-B), show that $\sin(90°-C) = \cos C$.
5. Find the value of sin 15° from sin (315° - 300°).
6. Verify that $\sin 90° = 2 \sin 45° \cos 45°$.
7. Find the value of $\sin(t + \pi)$.
8. Find the value of $\sin\frac{5}{12}\pi$ from $\sin\left(\frac{2}{3}\pi - \frac{\pi}{4}\right)$.

Compare your answers with the answers given at the end of this unit.

Sum and Difference of Tangent of two Angles

In the previous part we have been finding sine of the sum and difference of any two angles. Similarly you can find the compound formulae for tangent by using compound formulae for sine and cosines. The sum of tangent of two angles is obtained by taking the ratio of sum in sine of two angles to the sum in cosine of two angles.

tan (A + B) = $\frac{\text{Sin (A+B)}}{\text{Cos (A+B)}}$

$$\tan(A+B) = \frac{\sin A\cos B + \cos A\cos B}{\cos A\cos B - \sin A\sin B}$$

Dividing by cos A cos B throughout and simplify you obtain:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

For the difference of tangent of two you obtain by taking the ratio of difference in sines of two angles to the difference in cosines of two angles.

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$

$$\tan(A-B) = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$$

Dividing by cos A cos B throughout and simplify you obtain:

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Example

Without using table or calculator find the exact value of tan 15^0

Solution:

Tan 15^0 = tan (45^0 30^0)

But tan (A B) = $\frac{\text{tan A - tan B}}{\text{1+ tan A tan B}}$

$$\text{Tan } (45° - 30°) = \frac{\tan 45^0 - \tan 30^0}{1 + \tan 45^0 \tan 30^0}$$

For special angles

$\tan 45^0 = 1$

$\tan 30^0 = \sqrt{\frac{3}{3}}$

Then $\tan 15^0 = \dfrac{1 - \sqrt{\frac{3}{3}}}{1+1(\sqrt{\frac{3}{3}})}$

$= \dfrac{3 - \sqrt{3}}{3}$

$= \dfrac{3 + \sqrt{3}}{3}$

$\tan 15^0 = \dfrac{3 - \sqrt{3}}{3+\sqrt{3}}$

Activity 5

1. Given the triangles in figure 4.12 and 4.13, find tan (A+B)

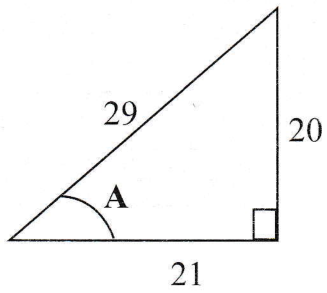


Figure 4.12

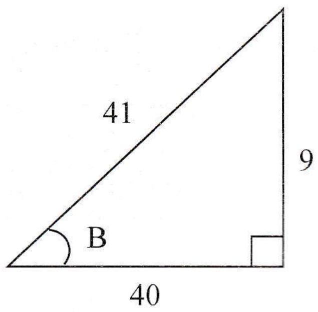


Figure 4.13

2. Let A be the angle for which sin A = 3/5 and cos A= 4/5. Find the exact value of tan (A+60).

3. Without using table as calculator find the exact value of tan 15°

Unit Reflection

Dear learner, you have already learnt how to derive and apply sine and cosine rules also the compound angles formulae for sine, cosine and tangent in solving trigonometric problems. How well can you apply this knowledge in various areas in your life? Have you ever visit land survey office? What do they apply the cosines sine rules and compound angles to subdivide the land into small plots? Do you think there is any other place this knowledge can be used?

Unit Assignment

1. Calculate the unknown sides and angles in figures 4.14 and 4.15.

(a)

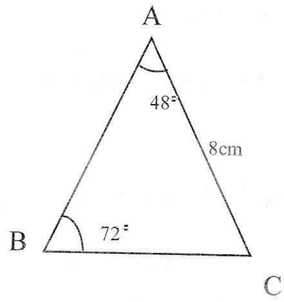


Figure 4.14

(b)

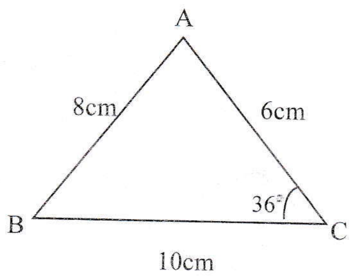


Figure 4.15

2. Given that a = 10cm, b = 8cm and c = 12cm from a triangle, find the angles in that triangle.
3. Find the unknown side and angles in each of the following:

(a)

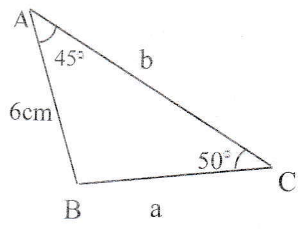


Figure 4.16

(b)

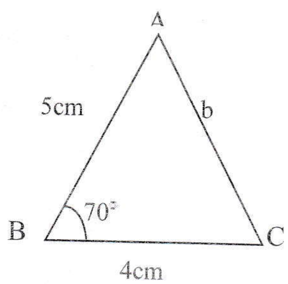


Figure 4.17

4. Show that sin 900 = 2sin 450 cos 450

5. Given that sin q° = sin 1500 and q° is an acute angle find the value of an angle q°.

6. Express the following in terms of the sine, cosine or tangent of an acute angle:
   (a) cos 200°.
   (b) sin 300°.
   (c) tan 240°.

7. Verify that for any angle C, cos (900 - C) = sin C.

8. Verify that $\cos\left(\frac{5}{6}\pi - \frac{5}{3}\pi\right) = \cos\frac{5}{6}\pi \cos\frac{5}{3}\pi + \sin\frac{5}{6}\pi \sin\frac{5}{3}\pi$

9. Find cos 2550 from cos (450 + 2100).

10. Verify that cos (1800 + 450) = cos 1800 cos 450 - sin 1800 sin 45o.

11. Verify that the following relations are true for all values of t
    (a) $\cos(\pi + t) = -\cos t$.
    (b) $\cos(2\pi + t) = \cos t$.
    (c) $\cos\left(\frac{\pi}{2} + t\right) = \sin t$.
    (d) $\cos\left(\frac{3}{2}\pi + t\right) = - \sin t$.

12. Verify that $\sin\left(\frac{3}{2}\pi - \frac{7}{6}\pi\right) = \sin\frac{3}{2}\pi \cos\frac{7}{6}\pi - \cos\frac{2}{3}\pi \sin\frac{7}{6}\pi$

13. By letting A = B in sin (A+B), find the formulae for sin 2A.

14. Find the sin 2250 from sin (1800 + 450

15. Verify that sin (1800+450 = sin1800cos 450 + cos 1800 sin 450

16. Without using table or calculator find the value of
    (a) $\tan 135^0$
    (b) $\tan 105^0$

17. Given the triangles in figures 4.18 and 4.19. Find the value of tan (C-D)

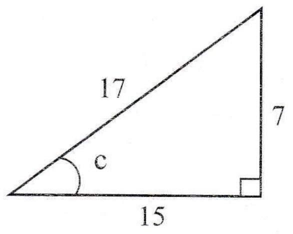


Figure 4.18

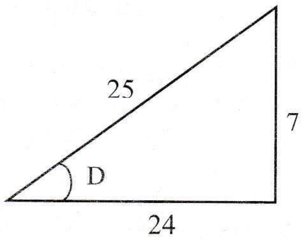


Figure 4.19

18. Verify that cos(180°-90°) = cos 180°cos 90° + sin180°sin 90°

19. Verify that cos(90° - 60°) = cos 90o cos 60° + sin 90°sin 60.°

Activities Answers

Activity 1

1. Side $b = 6.98$ *cm, angle A* 51.08^0, *angle B* $= 50.92^0$
2. Side $c = 17.56$ *cm, angle A* 36.46^0, *angle C* $= 33.32^0$
3. (a) Sides $a = 8.5$ *cm, angle c* 7.7^0, *angle A* $= 76^0$

   (b) Side $x = 121.1$ *cm, angle X* 128.8^0, $Z = 11.12^0$

Activity 2

1. angle A = 55°46′, angle B = 82°49′, angle C = 41°25′
2. angle A = 53°12′
3. (a) Side $a = 3.8$ *cm, angle B* 56^0, *angle A* $= 84^0$

   (b) Side $a = 7.53$ *cm, , angle A* $29^0 18'$, *angle C* $= 37^0 42'$

   (c) *angle A* 38^0, *angle B* $91^0 48'$, *angle C* $50^0 12'$

Activity 3

1. $\frac{\sqrt{2}+\sqrt{6}}{4}$

2. $\frac{\sqrt{2}-\sqrt{6}}{4}$

Activity 4

5. $\frac{\sqrt{6}-\sqrt{2}}{4}$

8. $\frac{\sqrt{6}+\sqrt{2}}{4}$

Activity 5

1. 0.0205
2. -8.1443
3. $\frac{3-\sqrt{3}}{3+\sqrt{3}}$

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